SENIOR THESIS

BROWN UNIVERSITY

Department of Physics

Triangulations, Polylogarithms, and Grassmannian Cluster Algebras in Particle Physics

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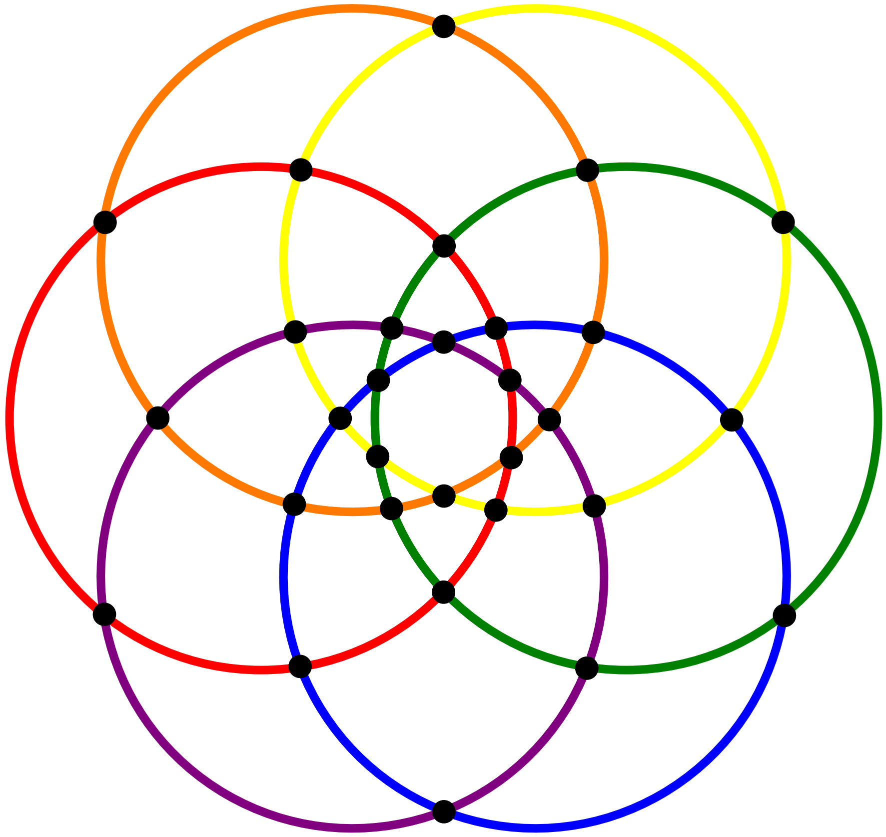

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Contents

1 Introduction

Recent work in the field of $\mathcal{N} = 4$ Super Yang-Mills scattering amplitudes has revealed a deep and unexpected connection between particle physics and cluster algebras. In particular, the combinatorial and algebraic properties of the cluster polylogarithms associated with the Grassmannian cluster algebra $\mathrm{Gr}(4, n)$ determine much of the structure of the planar n-particle MHV amplitude. Because cluster algebras themselves are still new, and cluster polylogarithms newer still, there are many open physical and mathematical questions about these structures. The results presented here answer some of these questions, and provide evidence and conjectures regarding others.

1.1 Acknowledgments

I would like to thank Professor Spradlin for his guidance, wisdom, and patience; Professor Volovich and Dr. Golden for many helpful conversations and clarifications; and Dan Parker for being an extraordinarily good colleague and friend.

2 Cluster Algebras

Cluster algebras are an extremely new mathematical structure, discovered in 2002 by Sergei Fomin and Andrei Zelevinsky[1]. In the past decade, they have found application in many areas of math and the mathematical sciences; the connection to $\mathcal{N} = 4$ SYM was uncovered in 2013.

The definition of a cluster algebra of rank n begins with a set of rational functions in n variables. These functions are gathered into subsets of size n, known as clusters. Terminology differs somewhat: the "cluster algebra" itself is either the set of clusters or the ring generated by the union of clusters (the set of $\mathbb{Q}$- or $\mathbb{Z}$-linear combinations of cluster coordinates; the underlying field is often left ambiguous). The set of clusters is the most important structure, so I will refer to that as a "cluster algebra" most of the time.

2.1 Definitions

- A **cluster** is a set C, whose elements are n $\mathcal{X}$-coordinates.
- An $\mathcal{X}$**-coordinate** is a rational functions in n variables $x_1, x_2, \ldots, x_n$. (We can also use $\mathcal{A}$-coordinates – see below – but these are less useful to physics.) It is often helpful to consider the $\mathcal{X}$-coordinate x in the context of an equivalence class that includes its reciprocal: $[x] = \{x, 1/x\}$. In general, the literature does not carefully distinguish x, $1/x$, and $[x]$.
- An **exchange function** is an antisymmetric integer function $b(x, y) = -b(y, x)$ on a cluster. It is often represented as an **exchange matrix** $[b]_{ij} = b(x_i, x_j)$, which requires that some ordering of the cluster be fixed. For our purposes, this is a Poisson bracket.
- A cluster algebra with an exchange function such that $|b(x, y)| \leq 1$ is called **simply laced**.
- A **seed** is a pair $S = (C, b)$ of a cluster of $\mathcal{X}$-coordinates and an exchange function on these coordinates. It is often represented as a **quiver** – a directed graph with vertex set C and with $b(x, y)$ edges from x to y. The notation $x \in S$ is shorthand for $x \in C$. The term "quiver" is also often used to refer to the unlabeled quiver (the graph with adjacency matrix $[b]$), or to the seed itself. A seed looks like this when depicted as a quiver:

  $$\boxed{x \longrightarrow y \longrightarrow z}$$

  This seed has the cluster $\{x, y, z\}$ and (under this ordering) the exchange matrix $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. Note that the exchange functions of all seeds must be compatible, so we can speak of a single exchange function (a partial function on pairs of $\mathcal{X}$-coordinates) for the entire cluster algebra.
- Two seeds (C, b) and (C', b') are (quiver-)**isomorphic** if there exists a bijection $f : C \to C'$ such that $b'(f(x), f(y)) = b(x, y)$ for all $x, y \in C$.

$$\begin{tikzcd}
& \frac{1}{y} \arrow[dl] & \\
x(1+y) \arrow[rr] & & \frac{zy}{1+y} \arrow[ul]
\end{tikzcd}$$

- Consider a seed $S = (C, b)$ and an element $x \in C$. We can follow a particular set of **mutation rules** to obtain a new seed $S' = (C', b')$ together with a bijection $\mu_x : C \to C'$ between their clusters. This process is called **mutation** of S on x. Mutation of $\boxed{x \to y \to z}$ on the coordinate y produces the quiver shown above.

  If some ordering of C is given, we can order C' via the bijection; then the exchange matrix $[b'] = \mu_i[b]$ depends only on $[b]$ and the index i of the mutated coordinate. (In other words, "quiver mutation" can be defined purely as a transformation of unlabeled directed graphs.)

- Two seeds S, S' are **mutation-equivalent** if there exists a sequence of mutations taking S to S'.

- A rank-n **cluster algebra** A_S is the set containing the ("initial") seed S, whose cluster is typically composed of identity functions $\{x_1, x_2, \ldots, x_n\}$, as well as all seeds mutation-equivalent to S.

- A cluster algebra is said to be of **finite type** if it has finitely many seeds. This quantity is known as the **order** $|A|$ of the cluster algebra.

- A cluster algebra has **finite mutation type** if it has finitely many seeds up to quiver isomorphism.

- The **cluster polytope** or **exchange graph** (many other names) of a cluster algebra is the (undirected) graph whose vertex set is all seeds mutation-equivalent to S, with an edge between seeds linked by a single mutation. We can associate these edges with the mutations μ_x and $\mu_{1/x}$ in each direction, or with the $\mathcal{X}$-coordinate equivalence class $[x]$. We sometimes choose to direct the edges of a polytope and associate it only with the mutation μ_x "in the same direction", or associate it with x only, excluding $1/x$.

2.1.1 Mutation Rules

Suppose we mutate a seed at x_k. The mutation rules for a quiver (in the form of the matrix $[b]$) are:

$${}'_{ij} = \begin{cases} -[b]_{ij} & k \in \{i, j\} \\ [b]_{ij} + [b]_{ik}[b]_{kj} & [b]_{ik} > 0, [b]_{kj} > 0 \\ [b]_{ij} - [b]_{ik}[b]_{kj} & [b]_{ik} < 0, [b]_{kj} < 0 \\ [b]_{ij} & [b]_{ik}[b]_{kj} \le 0. \end{cases} \tag{1}$$

There is a simpler way to remember this graphically:

- Reverse arrows to and from x_k
- For every path $x_i \to x_k \to x_j$, add the arrow $x_j \to x_i$ ("complete the triangle")
- Remove pairs of opposing arrows $a \rightleftarrows b$

Meanwhile, the mutation rules for the cluster variables x_i are:

$$x_i' = \begin{cases} x_i^{-1} & i = k \\ x_i \left(1 + x_k^{\operatorname{sgn}(b(x_i, x_k))}\right)^{b(x_i, x_k)} & i \neq k. \end{cases} \tag{2}$$

2.1.2 $\mathcal{A}$-Coordinates

In addition to the $\mathcal{X}$-coordinates, there is a second set of cluster coordinates known as $\mathcal{A}$-coordinates. The only difference is the mutation algorithm:

$$a_i' = \begin{cases} a_i^{-1}\left(\displaystyle\prod_{b(a_i,a_j)<0} a_j^{-b(a_i,a_j)} + \prod_{b(a_i,a_j)>0} a_j^{b(a_i,a_j)}\right) & i = k \\ a_i & i \neq k \end{cases} \tag{3}$$

This has a more geometric interpretation: in the case of simply-laced quivers (those for which $|b| \leq 1$), mutating a_k transforms it into the product of "incoming" $\mathcal{A}$-coordinates, plus the product of "outgoing" ones, divided by a_k. Unlike $\mathcal{X}$-coordinate mutation, this algorithm does not change any coordinate except the mutated one.

$\mathcal{A}$-coordinates were the original form of cluster coordinate and are still far more commonly used, but I will be more focused on $\mathcal{X}$-coordinates in this work.

There is a canonical map between $\mathcal{X}$- and $\mathcal{A}$-coordinates, given by

$$x_i = p(a_i) = \prod_j a_j^{b(a_i,a_j)} \tag{4}$$

This is compatible with mutation:

$$p(\mu_a(a')) = \mu_{p(a)}(p(a')) \tag{5}$$

As a result, applying p to an initial seed of $\mathcal{A}$-coordinates and $\mathcal{X}$-mutating repeatedly will produce the same cluster algebra that would be obtained by $\mathcal{A}$-mutating and then applying p to every seed after the fact.

2.1.3 Frozen Vertices

In an initial seed, it is common to designate some coordinates (or equivalently, quiver vertices) as **frozen**, meaning that they may not be mutated. The frozen status is preserved under mutation, so that freezing a coordinate decreases the rank of the cluster algebra by one. Because the isomorphism class of a cluster algebra depends only on the unfrozen coordinates, it may be more productive to think of frozen coordinates as being added on to the unfrozen seed. They are frequently called "coefficients" for this reason.

Frozen vertices are often depicted in rectangular boxes:

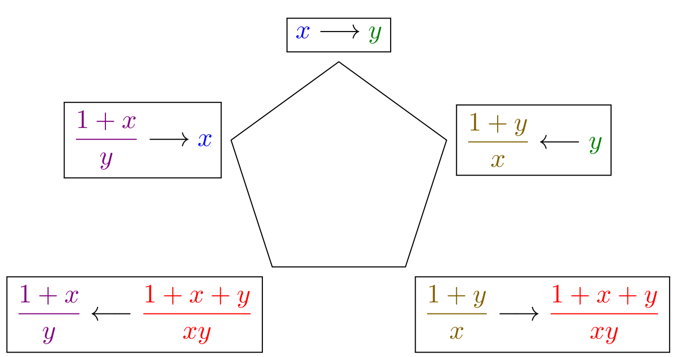
$$\boxed{x \longrightarrow y \longrightarrow z \longrightarrow \boxed{w}} \tag{6}$$

One very important fact is that p is not invertible: an initial $\mathcal{X}$-coordinate quiver can not always be obtained by applying p to some $\mathcal{A}$-coordinate quiver. If enough frozen vertices are added, however, a preimage under p can always be found.

We can strengthen the statement that freezing a vertex decreases the rank: cluster subalgebras are precisely the cluster algebras obtained by freezing various combinations of vertices.

2.1.4 Complete Example: A_2

This is the exchange graph of the cluster algebra A_2 with $\mathcal{A}$-coordinates shown:



(By the other meaning of "cluster algebra", A_2 is the ring generated by $\left\{x, y, \frac{1+y}{x}, \frac{1+x+y}{xy}, \frac{1+x}{y}\right\}$ over $\mathbb{Q}$ or $\mathbb{Z}$.)

2.2 Cartan-Killing Type Classification

One of the most important early theorems of cluster algebras is a classification theorem similar to that of semisimple Lie algebras. [2] Specifically, every finite-type irreducible cluster algebra contains some quiver shaped like a connected Dynkin diagram; every other finite-type cluster algebra is the product of some finite subset of these.

The irreducible cases include three infinite families:
A_n:

$$x_1$$

$$x_1 \longrightarrow x_2$$

$$x_1 \longrightarrow x_2 \longrightarrow x_3$$

$$x_1 \longrightarrow x_2 \longrightarrow x_3 \longrightarrow \cdots$$

B_n $(= C_n)$:

$$x_1 \rightrightarrows x_2$$

$$x_1 \rightrightarrows x_2 \longrightarrow x_3$$

$$x_1 \rightrightarrows x_2 \longrightarrow x_3 \longrightarrow \cdots$$

and D_n:

$$\begin{tikzcd} & x_1 \arrow[d] & \\ x_2 \arrow[r] & x_3 \arrow[r] & x_4 \end{tikzcd}$$

$$\begin{tikzcd} & x_1 \arrow[d] & & \\ x_2 \arrow[r] & x_3 \arrow[r] & x_4 \arrow[r] & x_5 \end{tikzcd}$$

$$\begin{tikzcd} & x_1 \arrow[d] & & & \\ x_2 \arrow[r] & x_3 \arrow[r] & x_4 \arrow[r] & x_5 \arrow[r] & \cdots \end{tikzcd}$$

There are also five "exceptional" cluster algebras that do not belong to an infinite family:

E_6:

$$\begin{tikzcd} & & x_1 \arrow[d] & & \\ x_2 \arrow[r] & x_3 \arrow[r] & x_4 \arrow[r] & x_5 \arrow[r] & x_6 \end{tikzcd}$$

E_7:

$$\begin{tikzcd}
& & x_1 \arrow[d] & & & \\
x_2 \arrow[r] & x_3 \arrow[r] & x_4 \arrow[r] & x_5 \arrow[r] & x_6 \arrow[r] & x_7
\end{tikzcd}$$

E_8:

$$\begin{tikzcd}
& & x_1 \arrow[d] & & & & \\
x_2 \arrow[r] & x_3 \arrow[r] & x_4 \arrow[r] & x_5 \arrow[r] & x_6 \arrow[r] & x_7 \arrow[r] & x_8
\end{tikzcd}$$

F_4:

$$x_1 \longrightarrow x_2 \rightrightarrows x_3 \longrightarrow x_4$$

and G_2:

$$x_1 \Rrightarrow x_2$$

Most of the time, we are concerned with finite-type simply-laced cluster algebras (A_n, D_n, E_n, and products thereof).

2.3 Rank and Order

2.3.1 Irreducible Simply-Laced Algebras

I have calculated that irreducible simply-laced cluster algebras of finite type (ADE cluster algebras) have distinct orders up to rank 300,000,000. I conjecture that this holds in general. Because there are only three E_n algebras, this reduces to the claim that

$$|A_n| \neq |D_m| \tag{7}$$

for all integers $n \geq 1, m \geq 4$. These orders are known[3][4]; they are related to the Catalan numbers,

$$C_n := \frac{1}{n+1}\binom{2n}{n} \tag{8}$$

In particular,

$$\begin{aligned} |A_n| =& C_{n+1} \\ |D_m| =& (3m-2)C_{m-1} \end{aligned} \tag{9}$$

So my conjecture is equivalent to the claim that $(3n+1)C_n$ is not a Catalan number for $n > 2$. (The case $n = 2$ corresponds to the exceptional isomorphism $D_3 \cong A_3$.) To make the conjecture more precise, let's define

$$N_k := \left[\frac{4^{k+3}}{3} - \frac{3(k+3)}{2}\right] \tag{10}$$

where $[x]$ is the nearest integer to x. The first term of N_k is derived from Stirling's approximation,

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \tag{11}$$

The second term of N_k is the result of trial and error.

Define k to be the integer such that $N_{k-1} < n \leq N_k$. Then I conjecture that, for $n \geq 4$,

$$|A_{n+k}| < |D_n| < |A_{n+k+1}| \tag{12}$$

Note, in particular, that k grows logarithmically with n.

In checking this conjecture, we only need to worry about the values $n = N_k$ where k increases by one: at these points, $|D_{N_k}|$ is slightly less than $|A_{N_k+1}|$, and $|D_{N_k+1}|$ is slightly more than $|A_{N_k+2}|$. For instance, when k increases from 0 to 1 at $N_0 = 17$, we have:

$$|A_{17}| < |D_{17}| < |A_{18}| < |A_{19}| < |D_{18}| < |A_{20}| \tag{13}$$

The relative difference between $|D_{17}|$ and $|A_{18}|$ is about 2%. The relative difference between $|D_{18}|$ and $|A_{19}|$ is about 3%.

The conjecture can be checked through large values of n very easily due to the slow growth of k; the limiting factor was Mathematica's inability to calculate $C_{357913918}$.

2.3.2 All Simply-Laced Algebras

Including reducible algebras, we quickly find coincidences such as:

$$|D_4| = |A_2 \times A_2 \times A_1| \tag{14}$$

Due to the fact that $10C_3 = C_3C_3C_2 = 50$.

The pair (rank,order) uniquely distinguishes simply-laced cluster algebras until rank 12, where we find three coincidences:

$$\begin{aligned} |A_7 \times A_4 \times A_1| = |D_5 \times A_5 \times A_2| &= 120120 \quad C_8C_5C_2 = 13C_4C_6C_3 \\ |A_7 \times A_5| = |D_8 \times A_1 \times A_1| &= 188760 \quad C_8C_6 = 22C_2C_2 \\ |A_9 \times A_3| = |A_{10} \times A_1 \times A_1| &= 235144 \quad C_{10}C_4 = C_{11}C_2C_2 \end{aligned} \tag{15}$$

Therefore, rank and order can be used to differentiate subalgebras easily for small rank, but not in general.

2.4 The Cluster Modular Group

The cluster modular group is described at length in [5].

Any isomorphism between two seeds in a cluster algebra can be extended uniquely to a permutation of all $\mathcal{X}-$coordinates in that algebra.

Let S, S' be two seeds of the algebra A, such that there is some quiver isomorphism $\varphi : S \to S'$. As stated earlier, quiver mutation depends only on quivers; thus, if two seeds are related by a quiver isomorphism φ, then mutating on x and $\varphi(x)$ produces a new pair of quiver-isomorphic seeds T, T'. In particular, we obtain a new quiver isomorphism $\varphi' : T \to T'$, where the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\mu_x"'] & T \arrow[d, "\mu_{\varphi(x)}"] \\ S' \arrow[r, "\varphi'"] & T' \end{tikzcd}$$

We can continue the process to obtain the full cluster algebras A_S and $A_{S'}$, with a bijection between the sets of seeds and a quiver isomorphism between each pair of bijected seeds. These algebras are of course equal to A as sets, so what we really have is a permutation of A. We can obtain an analogous permutation for every quiver isomorphism between seeds of A (including automorphisms of a single seed). These permutations form a group, known as the **cluster modular group** Γ of A. For each permutation $g \in \Gamma$ and seed $S \in A$, we have a quiver isomorphism $\varphi_{g,S} : S \to g(S)$. (Technically, Γ is the fundamental group of the groupoid consisting of these φ's.)

There is a very easy way to obtain the image of an $\mathcal{X}$-coordinate under an element $g \in \Gamma$, so long as we know the image of the initial seed (and so long as we use the "standard" initial cluster $\{x_1, x_2, \dots\}$). If S_0 is the initial seed, let $\varphi_{g,0}$ be the appropriate quiver isomorphism $S_0 \to g(S_0)$. Let $y(x_1, \dots, x_n)$ be some arbitrary $\mathcal{X}$-coordinate (note that this is a rational function of the initial cluster). Then

$$y \circ \varphi_{g,0} = \varphi_{g,S}(y) \tag{16}$$

This leads immediately to a mutation-free algorithm for computing the cluster modular group: find every seed with the same quiver as the initial seed, then construct the action of some element g of the cluster modular group by composing $\varphi_{g,0}$ with each seed of the cluster algebra.

Γ is also the group of symmetries of the polytope which preserve quivers. Let Γ' be the slightly larger group of symmetries that preserve quivers up to the orientation of arrows (up to the sign of b). Call it the **extended cluster modular group**. Then $\Gamma'/\Gamma \cong Z_2$; in other words, there is a Z_2 symmetry of the polytope that corresponds to arrow inversion and that is not contained in the cluster modular group. We have never encountered any symmetry of a cluster polytope that did not preserve quivers at least up to arrow-orientation, so I conjecture that the extended cluster modular group is the full symmetry group of the polytope.

One of the most useful applications of this information is in counting quiver shapes. Once we have found every seed in some quiver isomorphism class Q, we can immediately compute the order of Γ as the product of the number of automorphisms of the quiver and the number of seeds isomorphic to that quiver:

$$\begin{aligned} |\Gamma| =& |\mathrm{Aut}(Q)| \cdot |Q| \\ |\Gamma'| =& 2|\Gamma| \end{aligned} \tag{17}$$

(This is a consequence of the orbit-stabilizer theorem in group theory.)

From this, given any other quiver, we can count that quiver's symmetries and immediately deduce the number of seeds that will be quiver-isomorphic to it; this lets us understand the size and much of the geometric structure of a cluster algebra by cataloguing *quivers* rather than *seeds*, which requires a much simpler mutation algorithm. Furthermore, the automorphism group of every quiver will be a subgroup of Γ; technically, it is a stabilizer subgroup of the action of Γ on the cluster exchange graph.

3 Grassmannians and Cluster Algebras

3.1 Grassmannians

The **Grassmannian** $\mathrm{Gr}(k, n)$ is the space of k-complex-dimensional subspaces of $\mathbb{C}^n$ (often phrased as "k-planes in n-space", but it should be emphasized that these planes are through the origin.)

In the simplest case $k = 1$, this is just the complex projective space

$$\mathbb{CP}^{n-1} = \mathrm{Gr}(1, n) \tag{18}$$

There is a canonical isomorphism $\mathrm{Gr}(k, n) \cong \mathrm{Gr}(n-k, n)$, given by the orthogonal complement.

3.2 Plucker Coordinates and Relations

The most obvious way to represent a point in $\mathrm{Gr}(k, n)$ is by giving a basis for the subspace, consisting of k linearly independent points in $\mathbb{C}^n$. They are represented here as a matrix of row vectors for $k = 4, n = 6$:

$$(Z_1 Z_2 Z_3 Z_4 Z_5 Z_6) = \begin{pmatrix} z_{11} & z_{21} & z_{31} & z_{41} & z_{51} & z_{61} \\ z_{12} & z_{22} & z_{32} & z_{42} & z_{52} & z_{62} \\ z_{13} & z_{23} & z_{33} & z_{43} & z_{53} & z_{63} \\ z_{14} & z_{24} & z_{34} & z_{44} & z_{54} & z_{64} \end{pmatrix} \in \mathrm{Gr}(4, 6) \tag{19}$$

This is, however, highly nonunique; we can choose *any* basis, so we need to quotient out by invertible linear transformations ($\mathrm{GL}(k)$) to obtain a unique representation. We can achieve this by defining **Plücker coordinates**, which are simply the determinants of $k \times k$ matrix minors:

$$\langle ijk\ell\rangle := \det\left(Z_i Z_j Z_k Z_\ell\right) \tag{20}$$

Plücker coordinates do not depend on the particular basis we choose for our subspace, but they still uniquely identify a single subspace. They are somewhat overcomplete; we can write down **Plücker relations**, the simplest of which is

$$\langle Iij\rangle\langle Ipq\rangle = \langle Ipj\rangle\langle Iiq\rangle + \langle Iqj\rangle\langle Ipi\rangle \tag{21}$$

where I represents an initial sequence of indices which is the same for each coordinate.

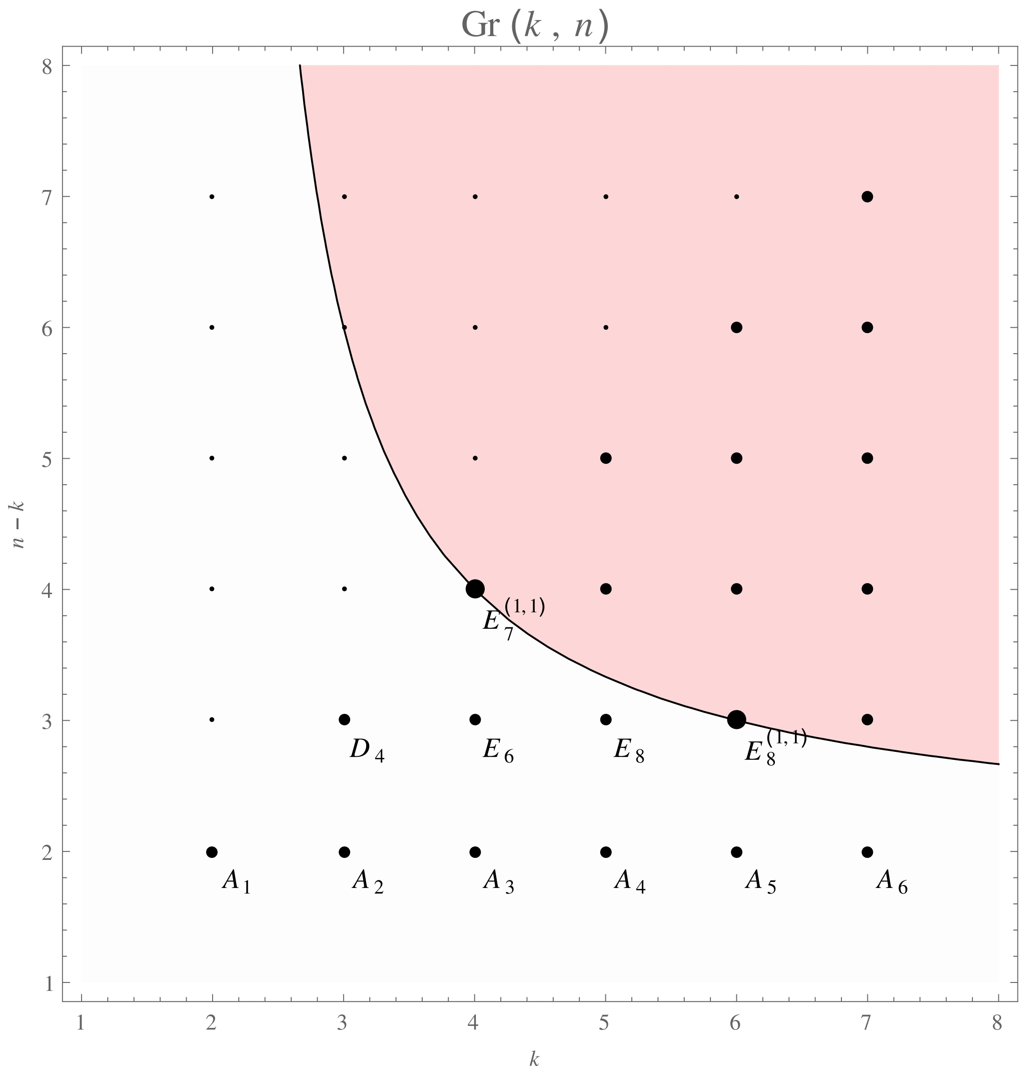
Crucially, this relation can be written as a mutation of $\mathcal{A}$-coordinates (mutating on $\langle Iij\rangle$):

$$\begin{tikzcd}
\boxed{\langle Ipj\rangle} \arrow[dr] & & \boxed{\langle Ipi\rangle} \\
\boxed{\langle Iiq\rangle} \arrow[r] & \langle Iij\rangle \arrow[ur] \arrow[r] \arrow[dd, Rightarrow] & \boxed{\langle Iqj\rangle} \\
& & \\
\boxed{\langle Ipj\rangle} \arrow[dr] & & \boxed{\langle Ipi\rangle} \\
\boxed{\langle Iiq\rangle} \arrow[r] & \langle Ipq\rangle \arrow[ur] \arrow[r] & \boxed{\langle Iqj\rangle}
\end{tikzcd}$$

3.3 The Cluster Algebra Gr(k, n)

In fact, each Grassmannian has an associated cluster algebra. An initial quiver for $\mathrm{Gr}(k, n)$ is given by the $(k-1)$-by-$(n-k-1)$ grid graph, with some frozen vertices. Initial $\mathcal{A}$-coordinates are given by the Plücker coordinates. A_n is therefore (isomorphic to) the $\mathrm{Gr}(2, n+3)$ cluster algebra. Plotting dimension k against codimension $(n-k)$, we can chart all Grassmannian cluster algebras.

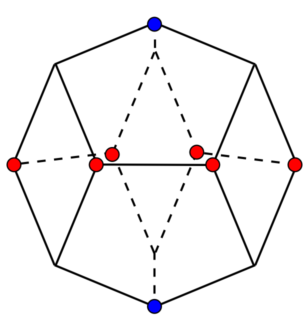
Note especially the hyperbolic boundary curve at $(k-2)(n-k-2) = 4$. Cluster algebras below the curve are of finite type; those above the line are of infinite mutation type; the three (two up to isomorphism) that fall on the curve have infinitely many seeds, but these seeds fall in a finite number of quiver isomorphism classes.



3.4 Grassmannian Cluster Modular Groups

3.4.1 A_3

A handy example of the properties of cluster modular groups is given by A_3 and its polytope. The "north pole" and "south pole" correspond to the oriented triangular quiver, which has a Z_3 automorphism group. There are six seeds of the form $a \to b \to c$, with no automorphisms. There are three seeds of the form $a \to b \leftarrow c$, with a Z_2 automorphism; there are also three of the form $a \leftarrow b \to c$, and these six form the "equator" of the A_3 polytope. In each case the product (# automorphisms)·(# seeds) is 6, the order of $\Gamma \cong Z_2 \times Z_3 \cong Z_6$. The A_3 polytope is shown below, with equator in red and poles in blue:



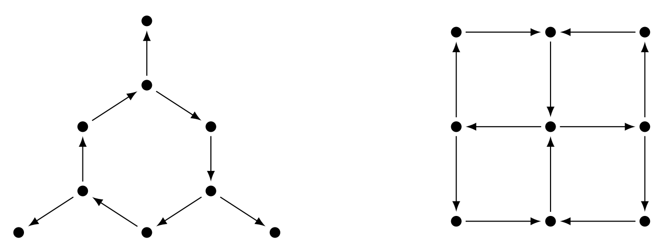
The full symmetry group of A_3, including arrow-reversals, is the semidirect product $D_{12} \cong Z_6 \rtimes Z_2$. This corresponds to the dihedral group of symmetries of the hexagon, with a Z_6 normal subgroup of rotations and a Z_2 reflection; in fact, the extended cluster modular group $\Gamma'(A_n) \cong D_{2n+6}$ is always the group of symmetries of the $n+3$-gon. The triangulation interpretation sheds light on this: rotating a triangulation doesn't alter the derived quiver, while reflecting it reverses the arrows.

There is also an independent "explanation" for the appearance of D_{12} here: A_3 is the Grassmannian cluster algebra $\mathrm{Gr}(2,6)$, and dihedral transformations of the six columns that can appear in Plucker coordinates give a D_{12} (possibly non-arrow-direction-preserving) symmetry. This is the more familiar source of dihedral symmetries in the context of scattering amplitudes, and corresponds to the dihedral symmetry of n-particle interactions in $\mathcal{N}=4$ SYM.

A_3 is also (by a small-n accident) the same Dynkin diagram as D_3, and illustrates a general property of the D_n cluster modular group: because the oriented n-cycle is a quiver of this algebra, $\Gamma(D_n)$ has Z_n as a subgroup and $\Gamma'(D_n)$ has D_{2n}. (Note that, confusingly, D_n is the name for a Dynkin diagram – hence a finite cluster algebra – and also happens to mean the dihedral group of order n.) The D_4 Dynkin diagram has a famous S_3 symmetry, which is the source of various "trialities" in mathematics. Here, this takes the form of an exceptional S_3 subgroup of $\Gamma(D_4)$, which is therefore nonabelian. (In contrast, while $\Gamma'(A_n)$ is nonabelian, $\Gamma(A_n)$ itself is abelian.)

Incidentally, we can generalize the notion of "poles" and "equators": the "poles" of a cluster algebra are the seeds (possibly more than two) which have orbits of size two under Γ, and the "equator" is the set of seeds whose stabilizer subgroup is Z_2 (equivalently, the set of seeds whose quivers have Z_2 automorphism group).

3.4.2 $E_7^{(1,1)}$



These two quivers of $\mathrm{Gr}(4,8) \cong E_7^{(1,1)}$ imply that that algebra's cluster modular group has Z_3 and Z_2^2 subgroups, and that the extended cluster modular group has D_6 and D_8 subgroups. Note that $\Gamma(E_7^{(1,1)})$ and $\Gamma'(E_7^{(1,1)})$ are both infinite.

In a finite cluster algebra, the first quiver would appear in 4/3 as many seeds as the second. We can conjecture that this should be the asymptotic ratio of frequencies of the quivers. In particular, consider the set of all seeds reachable in n mutations from the initial seed. Within this set, define r_n to be the number of seeds isomorphic to the first quiver divided by the number of seeds isomorphic to the second. We conjecture that $\lim_{n\to\infty} r_n = 4/3$.

Because $E_7^{(1,1)}$ is of finite mutation type, with 506 quiver isomorphism classes, the asymptotic density of each class should be nonzero. In principle, the densities (assuming this conjecture) would not be difficult to compute.

3.5 Cluster Algebras and Regular Tilings?

Grassmannian cluster algebras and regular tilings (and the symmetry groups of those tilings) obey similar classifications into finite, affine, and hyperbolic (or spherical, planar, and hyperbolic) cases.

In fact, one can construct identical tables of cluster algebras and tilings, as shown below.

Formally, we observe that **the cluster algebra Gr($p, p+q$) is finite iff the Coxeter group $[p,q]$ is finite, extended-affine iff $[p,q]$ is affine, and hyperbolic iff $[p,q]$ is hyperbolic.**

Or, in simpler language: **Gr($p, p+q$) is finite iff the tiling $\{p, q\}$ is spherical, infinite but of finite mutation type iff $\{p, q\}$ is planar, and of infinite mutation type iff $\{p, q\}$ is hyperbolic.**

This raises two questions:

- Is this previously known?
- Does this follow from a deeper connection between these structures, or is it coincidence?

If this is novel, and a simple proof can be found, it could provide a classification theorem for Grassmannian cluster algebras that does not rely on case analysis and computer search.

3.5.1 Grassmannian Cluster Algebras

Scott[6] and Fomin et al[7] have proven that the following classification of Grassmannian cluster algebras Gr($p, p+q$) (with subspace dimension p and codimension q) holds:

$p \backslash q$	2	3	4	5	6	7
2	A_1	A_2	A_3	A_4	A_5	A_6
3	A_2	D_4	E_6	E_8	$E_8^{(1,1)}$	Gr(3,10)
4	A_3	E_6	$E_7^{(1,1)}$	Gr(4,9)	Gr(4,10)	Gr(4,11)
5	A_4	E_8	Gr(5,9)	Gr(5,10)	Gr(5,11)	Gr(5,12)
6	A_5	$E_8^{(1,1)}$	Gr(6,10)	Gr(6,11)	Gr(6,12)	Gr(6,13)
7	A_6	Gr(7,10)	Gr(7,11)	Gr(7,12)	Gr(7,13)	Gr(7,14)

Green cells of the table denote finite cluster algebras, with finite Dynkin diagrams X_n.
Yellow cells are infinite, but with finite mutation type; these have extended affine Dynkin diagrams $X_n^{(1,1)}$.
Red cells are of infinite mutation type, with hyperbolic Dynkin diagrams.

This classification depends on the parameter $r = (p-2)(q-2)$ (see [7] Prop. 12.11):
Gr($p, p+q$) is finite for $r < 4$, infinite with finite mutation type for $r = 4$, and infinite mutation type for $r > 4$.

Fomin et al[7] prove this statement by considering individual cases.

There is a canonical isomorphism between Gr($p, p+q$) and Gr($q, q+p$).

3.5.2 Regular Tilings

This table depicts regular tilings $\{p, q\}$. These are two-dimensional surfaces formed by joining together p-gons, with q at each vertex.

Green cells are spherical tilings: hosohedra $\{2, q\}$, dihedra $\{p, 2\}$, and the five Platonic solids.
The three yellow cells are planar tilings.
The red cells are regular hyperbolic tilings.

It is easy to show that the spherical, planar, or hyperbolic nature of a tiling depends on $r = (p-2)(q-2)$, for the cases $r < 4, r = 4, r > 4$ respectively.

The tilings $\{p, q\}$ and $\{q, p\}$ are dual.

$_p\backslash^q$	2	3	4	5	6	7
2	{2,2}	{2,3}	{2,4}	{2,5}	{2,6}	{2,7}
3	{3,2}	tetrahedron	octahedron	icosahedron	triangular tiling	{3,7}
4	{4,2}	cube	square tiling	{4,5}	{4,6}	{4,7}
5	{5,2}	dodecahedron	{5,4}	{5,5}	{5,6}	{5,7}
6	{6,2}	hexagonal tiling	{6,4}	{6,5}	{6,6}	{6,7}
7	{7,2}	{7,3}	{7,4}	{7,5}	{7,6}	{7,7}

3.5.3 Coxeter Groups

There's another way to frame the above, which is more obviously connected to cluster algebras but less geometric. The tiling $\{p, q\}$ has the symmetry group $[p, q]$ in Coxeter notation; these have associated Dynkin diagrams and obey a Cartan-Killing classification.

$_p\backslash^q$	2	3	4	5	6	7
2	A_1^3	$A_1 \times A_2$	$A_1 \times BC_2$	$A_1 \times H_2$	$A_1 \times G_2$	$A_1 \times I_2(7)$
3	$A_2 \times A_1$	A_3	BC_3	H_3	$G_2^{(1)}$	[3,7]
4	$BC_2 \times A_1$	BC_3	$C_2^{(1)}$	[4,5]	[4,6]	[4,7]
5	$H_2 \times A_1$	H_3	[5,4]	[5,5]	[5,6]	[5,7]
6	$G_2 \times A_1$	$G_2^{(1)}$	[6,4]	[6,5]	[6,6]	[6,7]
7	$I_2(7) \times A_1$	[7,3]	[7,4]	[7,5]	[7,6]	[7,7]

Spherical tilings correspond to finite Coxeter groups X_n.
Planar tilings have affine Coxeter groups $X_n^{(1)}$ (also called $\widetilde{X}_n$).
Hyperbolic tilings have hyperbolic Coxeter groups.

Dual tilings have the same symmetries; $[p, q]$ and $[q, p]$ are isomorphic.

4 Polylogarithms and Iterated Integrals

The **classical polylogarithm** Li_k can be defined recursively:

$$\begin{aligned}\mathrm{Li}_1(z) &:= -\log(1-z) \\ \mathrm{Li}_k(z) &:= \int_0^z \mathrm{Li}_{k-1}(t)\mathrm{dlog}(t)\end{aligned} \tag{22}$$

Note that "classical polylogarithm" can also refer to any linear combination of products of Li_k functions. There is also a power series representation:

$$\text{Li}_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k} \tag{23}$$

The slightly more complicated **Goncharov polylogarithm** is defined in [8] as:

$$\begin{aligned} G(z) :=& 1 \\ G(a_1, \ldots, a_n; z) :=& \int_0^z G(a_2, \ldots, a_n; t) \text{dlog}(t - a_1) \end{aligned} \tag{24}$$

A **generalized polylogarithm** is, for our purposes, a linear combination of products of Goncharov polylogarithms.

4.1 Weight

The **weight** $w(f)$ (not standard notation) of a generalized polylogarithm f is defined recursively:

- $w(R) = 0$ where R is a rational function or constant.
- $w\left(\int f(t)\text{dlog}R(t)\right) = w(f) + 1$ where R is a rational function.
- $w(fg) = w(f) + w(g)$
- $w(f + g) = \max(w(f), w(g))$

In short, "weight" counts the number of integrations, and behaves much like the degree of a polynomial.

Whether transcendental constants have nonzero weight is inconsistent in the literature; π and multiple zeta values are sometimes treated as having nonzero weight, because they are special values of functions of those weights. Some authors have $w(c) = 0$ for any real constant c.

The space of weight-k polylogarithms, modulo products of functions of lower weight, is written $\mathcal{L}_k$. The subspace of classical polylogarithms is known as the Bloch group B_k. For $k = 1$, $\mathcal{L}_1 \cong \mathbb{C}^*$, the group of nonzero complex numbers. For $k = 2, 3$, $\mathcal{L}_k = B_k$. For higher k, there are non-classical polylogarithms. The Bloch group is usually defined in terms of

$$\{x\}_k := -\text{Li}_k(-x). \tag{25}$$

4.2 Symbol Arithmetic

Consider a generalized polylogarithm of weight k, with total differential

$$df = \sum_i g_i \text{dlog}R_k \tag{26}$$

where g_i are functions of weight $k - 1$. The **symbol** of f is a tensor product, defined recursively as[9]:

$$\mathcal{S}(f) := \sum_i \mathcal{S}(g_i) \otimes \log R_k \tag{27}$$

Like any tensor product, the symbol is multilinear, which interacts with the logarithm in interesting ways:

$$\begin{aligned} \log(PQ) \otimes \log R =& (\log P + \log Q) \otimes \log R \\ =& \log P \otimes \log R + \log Q \otimes \log R \\ \log 1 \otimes \log R =& 0 \otimes \log R \\ =& 0 \end{aligned} \tag{28}$$

However, the log sign is universally omitted, leading to the strange equations:

$$\begin{aligned} PQ \otimes R =& P \otimes R + Q \otimes R \\ 1 \otimes R =& 0 \end{aligned} \tag{29}$$

The second rule can be generalized to $c \otimes R = 0$ for any constant c.

The usefulness of the symbol depends on the crucial fact that it preserves highest-weight terms: if $\mathcal{S}(f) = \mathcal{S}(g)$, then f and g are of the same weight and $(f - g)$ is of lower weight. (The converse does not necessarily hold.) The symbol transforms functional identities into linear-algebraic ones.

Therefore, if we consider functions as representatives of elements of $\mathcal{L}_k$, the symbol map is injective. If we consider functions in their own right, with product terms included, the symbol map is non-injective.

4.3 The Coproduct

The **coproduct** of a function (technically, of a symbol) is the following map:

$$\delta(a_1 \otimes \ldots \otimes a_n) := \sum_{k=1}^{n-1} \rho\left(a_1 \otimes \ldots \otimes a_k\right) \bigwedge \rho\left(a_{k+1} \otimes \ldots \otimes a_n\right) \tag{30}$$

where ρ is the following projection map, which removes products of functions of lower weight to produce a unique representative of an element of $\mathcal{L}_k$:

$$\begin{aligned} \rho(a_1) &:= a_1 \\ \rho(a_1 \otimes \cdots \otimes a_k) &:= \frac{k-1}{k}\left[\rho(a_1 \otimes \cdots \otimes a_{k-1}) \otimes a_k - \rho(a_2 \otimes \cdots \otimes a_k) \otimes a_1\right] \end{aligned} \tag{31}$$

You can also consider this as a sum of maps $\delta = \sum_k \delta_{k,n-k}$, where $\delta_{k,n-k} : \mathcal{L}_n \to \mathcal{L}_k \wedge \mathcal{L}_{n-k}$ for $k \geq (n-k)$. (Technically, this wedge product should be a tensor product in the case of $n \neq (n-k)$, but the abuse of notation is standard.)

The most important fact about the coproduct is that $\delta^2 f = 0$ for any function f.

At weight $k > 3$, there are multiple possibilities for $\delta_{k,n-k}$. Together with the above fact, this allows us to construct non-classical polylogarithms from their coproducts. For instance, at weight 4, we have $\delta = \delta_{2,2} + \delta_{3,1}$. A unique weight-4 polylogarithmic function (up to products of functions of lower weight) can be reconstructed from the two coproduct components $f_{2,2} \in \Lambda^2 B_2$ and $f_{3,1} \in B_3 \otimes \mathbb{C}^*$, so long as $\delta(f_{2,2}) + \delta(f_{3,1}) = 0$. (Note that this is the condition $\delta^2 = 0$ in disguise.) Because $f_{2,2}$ and $f_{3,1}$ must be tensors of classical polylogarithms, restricting the entries in our symbols to some finite-dimensional space allows us to construct a related finite-dimensional space of non-classical weight-four polylogarithms. For the full details, see [10]. As we shall see, cluster algebras provide a very natural restriction of this kind, one which is useful for SYM amplitudes.

Classical polylogarithms have coproducts that lie entirely in $B_{k-1} \otimes \mathbb{C}^*$:

$$\begin{aligned} \delta\{x\}_2 &= (1+x) \bigwedge x \\ \delta\{x\}_k &= \{x\}_{k-1} \otimes x \quad (k > 2) \end{aligned} \tag{32}$$

It is conjectured that the converse holds. We will assume this conjecture in the following section.

4.4 Functions of the Form $G(a^k, b^\ell; z)$

Consider Goncharov polylogarithms of the form $G(a, \ldots, a, b, \ldots, b; z)$, hereafter abbreviated $G(a^k, b^l; z)$ or $G_{k,l}$. Dan Parker and I have proven that this function satisfies the coproduct criterion for classicality, and I suspect that it is the only (single-term) Goncharov polylogarithm that does.

4.4.1 The Symbol $S_{k,l}$

We claim that $G_{k,l}$ is classical; that is, the coproduct vanishes except possibly in its $\mathcal{L}_{k-1} \otimes \mathbb{C}^*$ component.

Denote the symbol $\mathcal{S}(G(a^k, b^l; z))$ by $S_{k,l}$. Then we have the following recursive formula:

$$S_{k,l} = \begin{cases} S_{k-1,0} \otimes \dfrac{a-z}{a} & l = 0 \\ S_{0,l-1} \otimes \dfrac{b-z}{b} & k = 0 \\ S_{k-1,l} \otimes \dfrac{a-z}{a-b} + S_{k,l-1} \otimes \dfrac{b-a}{b} & k,l > 0 \end{cases} \tag{33}$$

This formula allows us to produce all terms of $S_{k,l}$ by working backwards from the last entry in the symbol, decreasing either k or l until both are zero.

It also provides a useful bijection: each term can be mapped to a word containing the letters A and B, k and l times respectively. There is one term for each word, which can be written down as follows:

(1) Replace each of the initial run of consecutive identical letters (for example, the first three letters of AAABABBA) with $A_0 = \frac{a-z}{a}$ if they are A and $B_0 = \frac{b-z}{b}$ if they are B.

(2) Replace all other A's with $A_1 = \frac{a-z}{a-b}$ and B's with $B_1 = \frac{b-a}{b}$.

(3) Insert tensor products between entries.

Denote the set of all permutations of the word $\mathrm{A}^k\mathrm{B}^l$ by $W_{k,l}$. To give an example,

$$\begin{aligned} W_{3,2} =& \{\text{AAABB, AABAB, AABBA,} \\ & \text{ABAAB, ABABA, ABBAA,} \\ & \text{BAAAB, BAABA, BABAA,} \\ & \text{BBAAA}\} \\ S(a,a,a,b,b;z) = S_{3,2} =& A_0 \otimes A_0 \otimes A_0 \otimes B_1 \otimes B_1 + A_0 \otimes A_0 \otimes B_1 \otimes A_1 \otimes B_1 + A_0 \otimes A_0 \otimes B_1 \otimes B_1 \otimes A_1 \\ & + A_0 \otimes B_1 \otimes A_1 \otimes A_1 \otimes B_1 + A_0 \otimes B_1 \otimes A_1 \otimes B_1 \otimes A_1 + A_0 \otimes B_1 \otimes B_1 \otimes A_1 \otimes A_1 \\ & + B_0 \otimes A_1 \otimes A_1 \otimes A_1 \otimes B_1 + B_0 \otimes A_1 \otimes A_1 \otimes B_1 \otimes A_1 + B_0 \otimes A_1 \otimes B_1 \otimes A_1 \otimes A_1 \\ & + B_0 \otimes B_0 \otimes A_1 \otimes A_1 \otimes A_1 \end{aligned} \tag{34}$$

4.4.2 Proof of Classicality

We are now ready to show that $G_{k,l}$, of weight $w = k+l$, is classical; that is, that $\delta_{n,w-n} S_{k,l} = 0$ unless $n = 1$ or $w-1$.

For every term $t = a_1 \otimes a_2 \otimes \ldots a_w$, recall that

$$\delta_{n,w-n}(t) = \rho(a_1 \otimes \ldots \otimes a_n) \wedge \rho(a_{n+1} \otimes \ldots \otimes a_w) - \rho(a_{w-n+1} \otimes \ldots \otimes a_w) \wedge \rho(a_1 \otimes \ldots \otimes a_{w-n}) \tag{35}$$

Therefore, we also have (where a_{ij} denotes the i'th entry in the j'th term),

$$\delta_{n,w-n}(S_{k,l}) = \sum_j \rho\left(\bigotimes_{i=1}^{n} a_{ij}\right) \wedge \rho\left(\bigotimes_{i=n+1}^{w} a_{ij}\right) + \sum_j \rho\left(\bigotimes_{i=1}^{w-n} a_{ij}\right) \wedge \rho\left(\bigotimes_{i=w-n+1}^{w} a_{ij}\right) \tag{36}$$

Denoting the first of these two sums as T_n, we can write this as $\delta_{n,w-n}(S_{k,l}) = T_n + T_{w-n}$.

Claim: For $1 < n < (w-1)$, $T_n = 0$.

Note that, for symbols of weight $n > 1$, $\rho(a_1 \otimes a_2 \otimes \ldots \otimes a_n + \text{permutations}) = 0$. As a special case, $\rho(a \otimes a \otimes \ldots \otimes a) = 0$. These two identities will prove useful.

We can partition $W_{k,l}$ into two subsets: let one subset contain those whose initial run of identical letters is of length n or more, and let the other be its complement. We can divide the terms of T_n similarly:

$$T_n = X_n + Y_n \tag{37}$$

where X_n consists of the terms of T_n with at least n instances of either A_0 or B_0.

It is now easy to show that $X_n = 0$; the tensor product on the left side of the wedge product must be either $\bigotimes_{i=1}^n A_0$ or $\bigotimes_{i=1}^n B_0$, so its image under ρ is zero by the first of our identities.

For a term $\rho(L) \wedge \rho(R)$ of Y_n, note that the word in $W_{k,l}$ that corresponds to $L \otimes R$ has an "initial run" of length strictly less than n. Therefore, the tensor product R only contains the entries A_1 and B_1. Furthermore, Y_n will also contain all terms obtained by permuting the entries of R while keeping L fixed; this is because the partitioning of $W_{k,l}$ only depends on the first n letters in each word.

Therefore, we can write Y_n as a sum of terms of the form $\rho(L) \wedge \rho(R + \text{permutations})$. R is of length $(w-n) > 1$, so these terms are all zero by the second of our identities.

Hence $T_n = 0$ and $\delta_{n,w-n}(S_{k,l}) = 0$, so $G_{k,l}$ is a purely classical function.

4.4.3 The Reverse Direction

I conjecture that the functions $G_{k,l}$ are the only *purely* classical Goncharov polylogarithms. Note, however that there are many classical polylogarithms (by the coproduct criterion) that are sums of several different Goncharov polylogarithms, which generally are not of this form.

5 Cluster Polylogarithms

Cluster algebras have several associated spaces of polylogarithms, known as cluster polylogarithms. These spaces are motivated by a cluster-algebraic structure found (more or less empirically) in well-known polylogarithmic identities.

5.1 The Abel and Goncharov Identities

Recall that the cluster $\mathcal{A}$-coordinates of A_2 are given by $\left\{x, y, \frac{1+y}{x}, \frac{1+x+y}{xy}, \frac{1+x}{y}\right\}$, with clusters consisting of successive pairs of these. The following identity holds:

$$\begin{aligned}&\mathrm{Li}_2(-x) + \mathrm{Li}_2(-y) + \mathrm{Li}_2\left(-\tfrac{1+y}{x}\right) + \mathrm{Li}_2\left(-\tfrac{1+x+y}{xy}\right) + \mathrm{Li}_2\left(-\tfrac{1+x}{y}\right)\\ &+ \log x \log y + \log y \log\left(\tfrac{1+y}{x}\right) + \log\left(\tfrac{1+y}{x}\right)\log\left(\tfrac{1+x+y}{xy}\right) + \log\left(\tfrac{1+x+y}{xy}\right)\log\left(\tfrac{1+x}{y}\right) + \log\left(\tfrac{1+x}{y}\right)\log x = -\frac{\pi^2}{2}\end{aligned}$$

More concisely, we can write the $\mathcal{A}$-coordinates as $\{x_1, x_2, x_3, x_4, x_5\}$, so that

$$\sum_i \left[\mathrm{Li}_2(-x_i) + \log x_i \log x_{i+1}\right] + \frac{\pi^2}{2} = 0 \tag{38}$$

Note that there is a Li_2 term for each coordinate and a $\log\log$ term for each cluster. (The latter, being products of functions of lower weight, are generally ignored in this line of work; I calculated them on a whim and was surprised by the additional "clustery" structure.) This identity, in a somewhat different form, was discovered by Niels Abel in the nineteenth century; it is known as the Abel identity, A_2 identity, or pentagon identity.

An unexpected 40-term identity for Li_3 was discovered recently by Goncharov et al[11]. It is similar in flavor to the Abel identity. Just as the Abel identity can be understood in terms of the coordinates of the cluster algebra A_2, the Goncharov identity can be understood in terms of the coordinates of D_4. This identity is also known as the D_4 identity or simply "the forty-term trilogarithm identity."

5.2 $\mathcal{X}$-Functions and $\mathcal{A}$-Functions

We will now cover the two principle types of cluster polylogarithms, as seen in [10]. A **cluster $\mathcal{A}$ function** of the cluster algebra A is a conformally-invariant function f of weight k whose symbol is of the form

$$\mathcal{S}(f) = \sum_i c_i \left(a_{i1} \otimes \cdots \otimes a_{ik}\right) \tag{39}$$

where a_{ij} is a cluster $\mathcal{A}$-coordinate on A. Conformal invariance, in the case of Grassmannian cluster algebras, means that the function is invariant under the map $Z_i \mapsto t_i Z_i$. It is conjectured that this definition is equivalent to saying that the symbol of f can be written in the form

$$\mathcal{S}(f) = \sum_i c_i \left(x_{i1} \otimes \cdots \otimes x_{ik}\right) \tag{40}$$

where x_{ij} is a cluster $\mathcal{X}$-coordinate on A.

A **cluster $\mathcal{X}$ function** (often simply "cluster function") of weight $k < 4$ is a function of the form

$$\sum_i c_i \mathrm{Li}_k(-x_i) \tag{41}$$

where x_i is an $\mathcal{X}$-coordinate on A. At weight 4, a cluster $\mathcal{X}$-function is a cluster $\mathcal{A}$-function f whose *coproduct* is of the form

$$\sum_i c_i \left(\{x_{i1}\}_2 \wedge \{x_{i2}\}_2\right) + \sum_j d_j \left(\{x_{j1}\}_3 \otimes x_{j2}\right) \tag{42}$$

5.3 The Pentagon Function

There is only one known non-classical cluster $\mathcal{X}$-function: the A_2 function. This "function" is generally thought of as an element of $\mathcal{L}_4$ modulo B_4, so it has many possible functional representatives; its $\Lambda^2 B_2$ coproduct, however, is unique. The unique skew-dihedral invariant coproduct of f_{A_2} is[10]:

$$\begin{aligned}\delta_{2,2}(f_{A_2}) &= \sum_{i,j=1}^{5} j\{x_i\}_2 \wedge \{x_{i+j}\}_2 \\ \delta_{3,1}(f_{A_2}) &= 5\sum_{i=1}^{5} \left(\{x_{i+1}\}_3 \otimes x_i - \{x_i\}_3 \otimes x_{i+1}\right)\end{aligned} \tag{43}$$

It is immediately clear that, by the coproduct classicality criterion, this is indeed a nonclassical cluster function.

5.4 Results and Conjectures

We have built up a collection of conjectures – some new, some old – regarding the nature of the cluster $\mathcal{X}$-functions that appear at each weight for each cluster algebra, as well as the identities they obey. These have been tested on various finite-type Grassmannian cluster algebras.

5.4.1 Weight 2

The only weight-2 cluster function is Li_2. This function obeys the Abel pentagon identity; I conjecture that this is the only linear relation, in the sense that any linear relation between Li_2's of a cluster algebra can be decomposed into a sum of the Abel identities associated with A_2 subalgebras.

5.4.2 Weight 3

The only weight-3 function is Li_3. This obeys Goncharov's famous D_4 trilogarithm identity; I conjecture that this is again the only linear relation.

5.4.3 Weight 4

We conjecture that the only weight-4 functions are Li_4 and f_{A_2}. Dan and I have confirmed this for E_6 (see [10], footnote 8) and many A_n.

There are no known Li_4 identities. We conjecture that the only f_{A_2} identities are the two identities found in D_4. I have confirmed this for E_6.

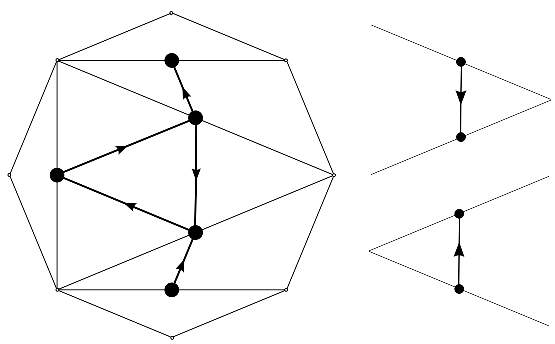
These D_4 identities have an interesting geometric structure. The 36 A_2 subalgebras, and hence 36 pentagon functions, of D_4 can be partitioned into three sets of 12. The pentagons are related geometrically by a Z_3 symmetry of the polytope which derives from the threefold symmetry of the D_4 Dynkin diagram. The sum of the pentagon functions in each set is the same.

6 A_n Cluster Algebras and Polygon Triangulations

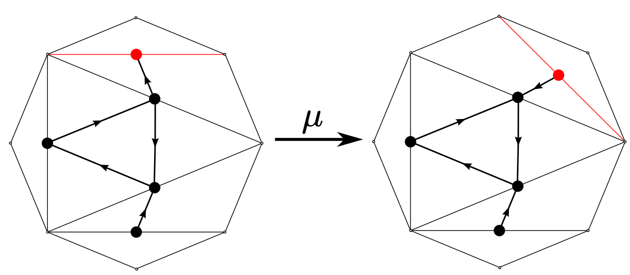
6.1 Triangulations

The A_n algebras have an interpretation in terms of $(n+3)$-gon triangulations.

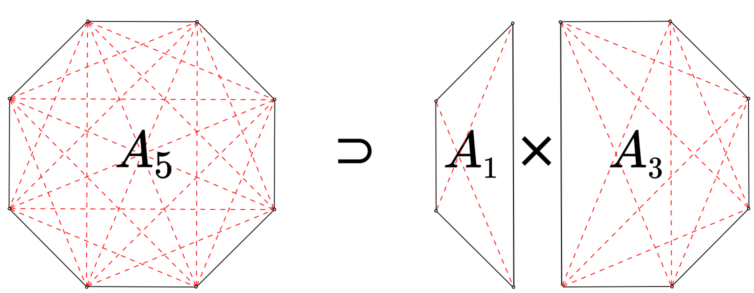
Each seed (and its quiver) corresponds to a triangulation. Quiver vertices correspond to internal chords of the triangulation. Vertices connected by an edge correspond to chords that are adjacent at a corner of the triangulation. If we overlay the quiver on the triangulation, quiver edges are oriented counterclockwise relative to the shared corner of the chords.



Mutation corresponds to a chord-flip or **Whitehead move**:



Subalgebras can be obtained, as always, by freezing a vertex; here, that means freezing a chord of a triangulation. In the case that this cuts off a triangle from the rest of the figure, we are left with triangulations of an $(n+2)$-gon and hence an A_{n-1} subalgebra. More generally, if a $k+3$-gon is on one side of the frozen chord and an $(n-k+2)$-gon on the other, we have a subalgebra isomorphic to $A_k \times A_{n-k-1}$.:



Through this connection to triangulations, the A_n cluster polytope has been known in the combinatorial literature for decades (under the name "Stasheff polytope" or "associahedron"). Triangulations of the $(n+3)$-gon can also be mapped to parenthetical groupings of $(n+2)$ letters (hence "associahedron"), or to binary trees, or various other combinatorial objects enumerated by Catalan numbers.

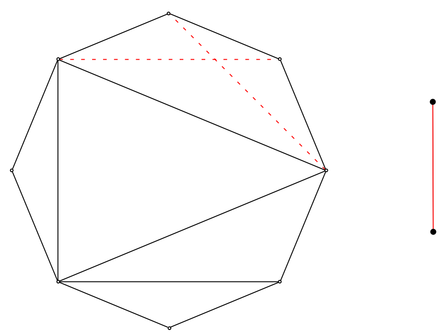
6.2 A_k Functions and Partial Triangulations

There is a well-known bijective correspondence between subalgebras of A_n and partial triangulations of an (n+3)-gon. For instance, complete triangulations of an octagon correspond to clusters of A_5, and triangulations of a hexagon correspond to clusters of A_3. An A_3 subalgebra of A_5 corresponds to a partial triangulation of an octagon that leaves a hexagon untriangulated; triangulating this hexagon then gives a particular cluster of this subalgebra. In the diagram above, we can imagine freezing an additional chord to remove the A_1 factor from the $A_1 \times A_3$ subalgebra.

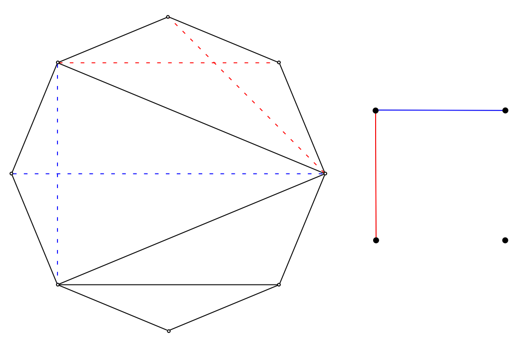
We are concerned mostly with A_1 and A_2 subalgebras, because these generate $\mathcal{X}$-coordinates and pentagon functions respectively.

6.2.1 Counting $\mathcal{X}$-coordinates

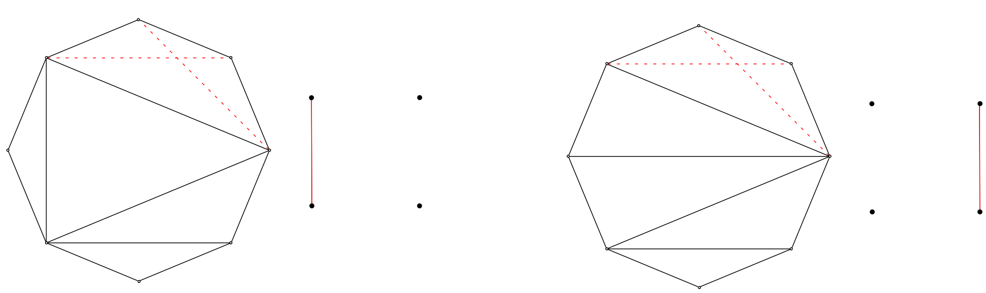
An edge of a cluster polytope (an A_1 subalgebra) corresponds to two triangulations that are related by a chord-flip:



In other words, we can view edges as partial triangulations of the polygon that contain a single untriangulated quadrilateral. Two edges lie adjacent to each other on an $A_1 \times A_1$ square (in the cluster polytope) if the corresponding quadrilaterals in the polygon do not overlap; that is, if the chord-flips commute:

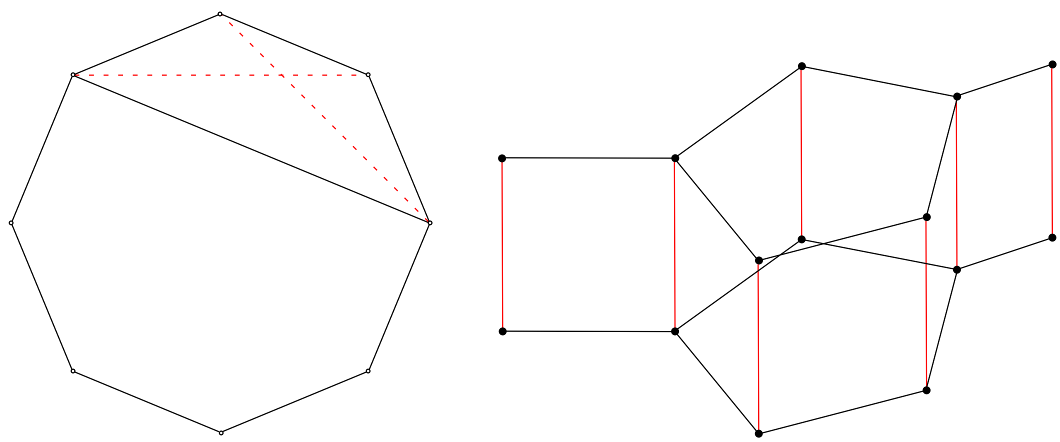


Two A_1's opposite each other on a square correspond to triangulations in which the same quadrilateral is left untriangulated, differing only by a single chord-flip outside this quadrilateral:



Recall that we can associate an edge of the polytope with a mutation μ_x or with the $\mathcal{X}$-coordinate equivalence class $[x]$, which we will call an $\mathcal{X}$-coordinate for brevity.

In general, two edges in the cluster polytope share an $\mathcal{X}$-coordinate if they are connected by a sequence of squares. In terms of the polygon, these edges correspond to two partial triangulations, with the same quadrilateral untriangulated, differing by some sequence of chord-flips outside the quadrilateral. The set of all edges with the same $\mathcal{X}$-coordinate – and the $\mathcal{X}$-coordinate itself – can be put into correspondence with the quadrilateral itself:

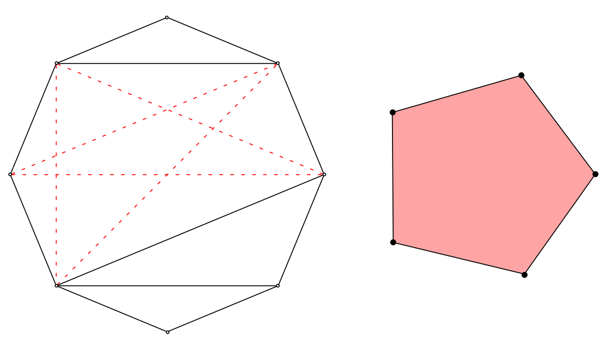


The highlighted edges above all share an $\mathcal{X}$-coordinate. Therefore, the number of distinct $\mathcal{X}$-coordinates of A_n, up to reciprocals, is given by the number of quadrilaterals with vertices on the (n+3)-gon, which is simply

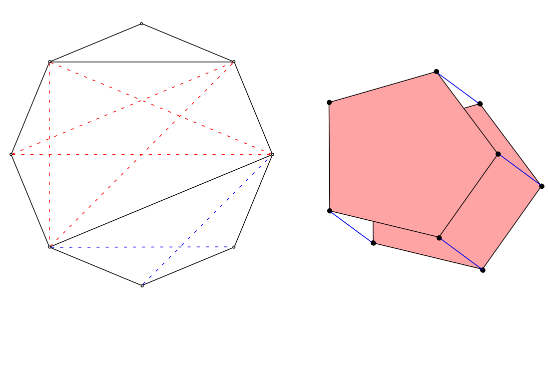
$$\binom{n+3}{4} \tag{44}$$

6.2.2 Counting A_2 Functions

A_2 functions work analogously. A partial triangulation with a pentagon left untriangulated corresponds to an A_2 subalgebra and an A_2 function:



An $A_1 \times A_2$ subalgebra appears in the polygon as a partial triangulation containing a pentagon and a quadrilateral; in the cluster polytope, it appears as two pentagons connected in a pentaprism:



Due to the squares involved, the pentagons have the same $\mathcal{X}$-coordinates and hence the same A_2 function. We can proceed as we did above for individual $\mathcal{X}$-coordinates. The set of pentagons that produce a particular A_2 function corresponds to the set of those partial triangulations of the (n+3)-gon in which a particular pentagon is left untriangulated. We can associate it with the untriangulated pentagon itself. This generalizes to a bijection between distinct A_2 functions and pentagons in the $(n+3)$-gon. Therefore, the number of distinct A_2 functions in A_n is given by:

$$\binom{n+3}{5} \tag{45}$$

...assuming that the Abel identities do not ever add up to exact equalities between otherwise-distinct A_2 functions; this assumption has held as far as A_8.

6.2.3 Generalization

Let $N = |A_k|$. Let the "A_k-function" $f_k(X_i)$ be any function that depends on the $\mathcal{X}$-coordinates $X_i = \{x_{i,1}, \ldots, x_{i,N}\}$ of the i-th A_k subalgebra of some cluster algebra. Assume that f_k is sufficiently complicated that $f_k(X_i) = \pm f_k(X_j)$ only if $X_i = X_j$ as a set. Empirically, this is apparently true for A_3 functions, A_2 functions, and "A_1 functions" (the logarithms of individual $\mathcal{X}$-coordinates).
Then the number of distinct "A_k-functions" in A_n is given by

$$\binom{n+3}{k+3} \tag{46}$$

Oddly, this implies that for $n \geq 12$, there are more distinct A_3 functions than A_2 functions. This is counterintuitive, but nothing forbids it. (On the other hand, there can never be more *linearly-independent* A_3 functions than A_2 functions.)

6.2.4 An Interesting Identity

There are no Li_4 identities on A_n, at least as far as A_8. Therefore, the total number of weight-4 cluster $\mathcal{X}$-functions on A_n is the number of $\mathcal{X}$-coordinates plus the number of distinct A_2 functions:

$$\binom{n+3}{4} + \binom{n+3}{5} = \binom{n+4}{5} \tag{47}$$

This is the same as the number of A_2 functions on A_{n+1}. That is, assuming A_2 functions are the only non-classical cluster functions at weight 4,

$$\dim(\mathcal{L}_4(A_n)) = \dim\left(\mathcal{L}_4(A_{n+1})/\text{B}_4(A_{n+1})\right) \tag{48}$$

This coincidence between A_2 functions of one weight and total weight-4 functions of the next lower weight was first noticed by Dan Parker.

6.3 Hedgehog Bases for A_n

A simple cluster-geometric construction provides a basis for cluster $\mathcal{A}$-functions on A_n. This work was done with Dan Parker and derives heavily from the results of Francis Brown[12].

Given two $\mathcal{X}$-coordinates x_i and x_j, we say that $x_i \sim x_j$ if their difference can be expressed as a product of $\mathcal{X}$-coordinates,

$$x_i - x_j = \pm \prod_\ell x_\ell^{m_\ell} \tag{49}$$

where x_ℓ is an $\mathcal{X}$-coordinate and $m_\ell \in \mathbb{Z}$. We can form a graph whose vertices are $\mathcal{X}$-coordinates, with an edge between x_i and x_j if $x_i \sim x_j$; see the cover illustration for an example on A_3. (Note that x and $1/x$ are distinct $\mathcal{X}$-coordinates in this context.) Complete subgraphs with n vertices are referred to as n-cliques. More precisely, let $q = \{q_1, \ldots, q_n\}$ be a set of n $\mathcal{X}$-coordinates. q is an n-**clique** if $q_i \sim q_j$ for all pairs $q_i, q_j \in q$.

Given an n-clique q in A_n, define the k-**Goncharov clique basis** of q as follows:

$$GCB_k(q) := G(\text{Lyn}_k\{0,1\}; -q_1) \cup G(\text{Lyn}_k\{0,1,-q_1\}; -q_2) \cup \cdots \cup G(\text{Lyn}_k\{0,1,-q_1,\ldots,-q_{n-1}\}; -q_n) \tag{50}$$

$G(\mathrm{Lyn}_k(A); z)$ is shorthand for the set of Goncharov polylogarithms whose arguments (except for z) form a Lyndon word of length k in the alphabet A.
Then it is known (mostly from Brown's work) that $GCB_k(q)$ is a basis for weight k cluster $\mathcal{X}$-functions on A_n.

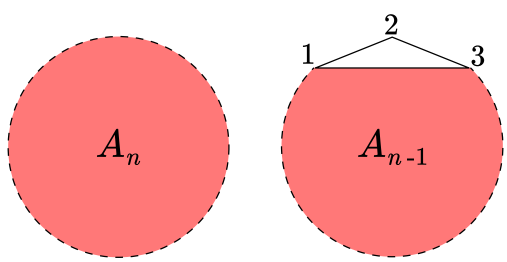
Dan and I have found a simple method for finding n-cliques of A_n, which we have dubbed **hedgehogs** of A_{n-1} subalgebras.

Let $B \cong A_{n-1}$ be a subalgebra of A_n. For each seed $S \subset B$, there will be a unique $\mathcal{X}$-coordinate x such that $\mu_x(S) \notin B$. The collection of each such x, for every seed of B, is the hedgehog of B. Equivalently, the hedgehog is the set of $\mathcal{X}$-coordinates x of B such that $1/x$ is not an $\mathcal{X}$-coordinate of B.

This construction is easier to understand graphically. Recall that each edge of a cluster polytope is associated with a mutation $S \to \mu_x(S)$, or equivalently $\mu_{1/x}(S') \to S'$. Orient the edge from S to S', and associate it with x; in the reverse orientation, associate it with $1/x$.

With this association in mind, the hedgehog of B is the set of oriented edges pointing outward from each seed of B into the rest of A_n, much like the spines of a hedgehog.

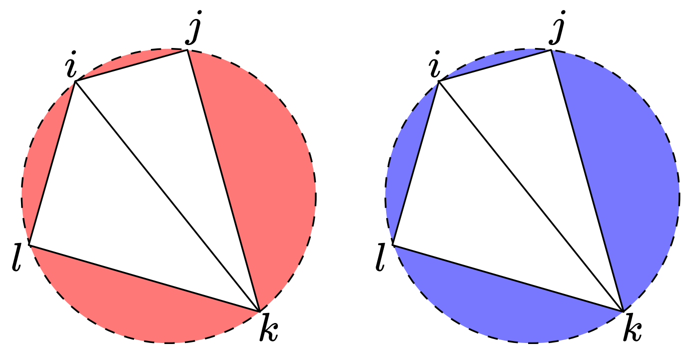
A final equivalent way to define hedgehogs is through polygon triangulations. Recall that seeds of A_n can be identified with triangulations of an $(n+3)$-gon; each A_{n-1} subalgebra corresponds to the set of triangulations that fix some $(n+2)$-gon. Restricting to one of these subalgebras corresponds to fixing an "almost external" triangulation chord: an internal chord that separates a triangle from an $(n+2)$-gon.
An immediate consequence is that A_n has $(n+3)$ subalgebras isomorphic to A_{n-1}. Recall that individual $\mathcal{X}$-coordinates in a seed can be associated with chords of the corresponding triangulation. The hedgehog of an A_{n-1} subalgebra is the set of $\mathcal{X}$-coordinates corresponding to a particular "almost external" chord in every triangulation in which it appears. The chord (13) below is "almost external":



6.3.1 The Hedgehog Theorem

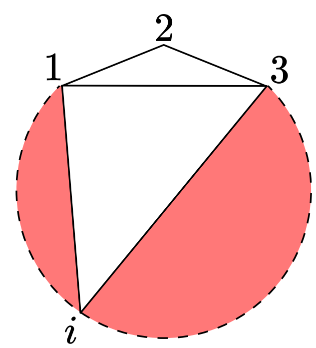
Every hedgehog in A_n is an n-clique. (We conjecture that the converse holds: there are no other n-cliques, except for the **reciprocal hedgehogs** $q^{-1} = \{q_i^{-1} | q_i \in q\}$, where q is a hedgehog.)

It is easiest to prove this in the triangulation picture. Flipping a chord E corresponds to mutation on that coordinate, sending $x \mapsto 1/x$; flipping a different chord that shares a triangle with E corresponds to a mutation that affects x; flipping a "commuting" chord (one that does not share a triangle with E) corresponds to a mutation that does not affect x. More precisely, chords in two different triangulations correspond to the same $\mathcal{X}$-coordinate (in two different seeds) iff the chords span the same quadrilateral:

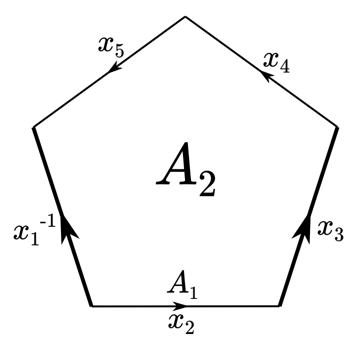


Here, the red and blue regions represent different subtriangulations. The chords (ik) have the same $\mathcal{X}$-coordinate. This is true because flipping a chord will alter this quadrilateral iff the corresponding mutation changes x.

When we choose a subalgebra for our hedgehog, we fix three vertices $\{1, 2, 3\}$ out of $n + 3$. The $\mathcal{X}$-coordinates in the hedgehog will correspond to the chord (13) in each triangulation; there are n vertices of the polygon left, so there are n possible quadrilaterals $(123i)$ and hence n distinct $\mathcal{X}$-coordinates in the hedgehog.



We can show that these form a clique by using a useful lemma: in A_2 (and therefore in any A_2 subalgebra), the variables $\{x_1^{-1}, x_3\}$ form a 2-clique. This is just the hedgehog theorem for $n = 2$, where x_1^{-1} and x_3 form the hedgehog for the A_1 subalgebra (edge) corresponding to x_2:

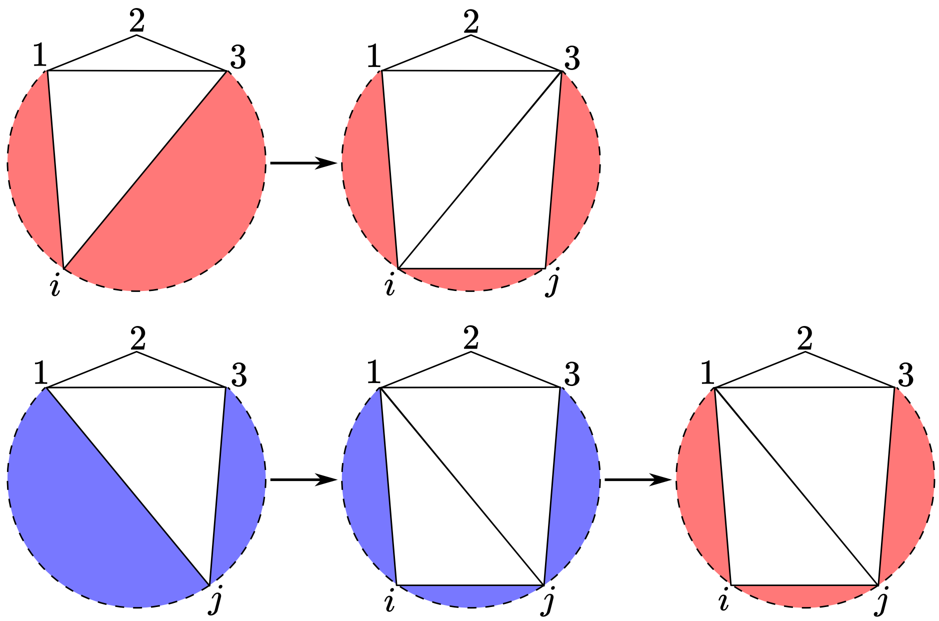


Note the directions of the arrows. The proof is straightforward:

$$
\begin{aligned}
x_3 =& \frac{1 + x_2}{x_1} \\
x_1^{-1} - x_3 =& \frac{1}{x_1} - \frac{1 + x_2}{x_1} = -\frac{x_2}{x_1} = -x_1 x_2^{-1}
\end{aligned}
\tag{51}
$$

Let $\{x_1, x_2, x_3, x_4, x_5\}$ be the $\mathcal{X}$-coordinates of any A_2 subalgebra. Define x_1^{-1} and x_3 to be **pentagon separated** (this term is motivated by the cluster-polytope picture). In the triangulation picture, pentagon-separated $\mathcal{X}$-coordinates x, y are associated with the same chord in two triangulations, which differ by a single flip of a "non-commuting" chord.

The only step that remains is to show that any two triangulations of the same A_{n-1} subalgebra can be mutated, without altering the crucial quadrilateral, so that they differ by a single chord-flip. The proof of this is entirely visual, again letting different colors stand for distinct arbitrary subtriangulations:



7 $\mathcal{N} = 4$ SYM and Cluster Polylogarithms

$\mathcal{N} = 4$ Super Yang-Mills is a toy model of physics. It is a quantum field theory (QFT) and a gauge theory, like the Standard Model, but calculations are far simpler to carry out. Because $\mathcal{N} = 4$ SYM is conformal, it can be related to quantum gravity through the AdS-CFT correspondence. Because it is maximally supersymmetric, it is far more unique and constrained than the Standard Model.

The primary object of interest in any QFT is the scattering amplitude. This is a complex function of n variables, each representing the momentum of an incoming or outgoing particle in an interaction; the squared magnitude of the amplitude provides a probability density for that particular interaction to happen in that particular configuration.

We are interested in one of the simplest kinds of scattering amplitudes in $\mathcal{N} = 4$ SYM, the planar MHV remainder function for n gluons. Configurations of n momenta can be represented as n twistor variables Z_i; together, the entire configuration is an element of $\mathrm{Gr}(4, n)$. The amplitude itself, when approximated to ℓ loops, is conjectured to be a weight-2ℓ generalized polylogarithm whose arguments are conformal cross-ratios of Plücker coordinates. It is further conjectured to be a cluster $\mathcal{A}$-function on the Grassmannian cluster algebra associated with $\mathrm{Gr}(4, n)$. For more details, see [11].

A good example of cluster structure appearing in SYM amplitudes is the following rule:

Two-loop MHV amplitudes in $\mathcal{N} = 4$ SYM have $\Lambda^2 B_2$ coproducts that can be expressed as a linear combination of terms $\{x\}_2 \wedge \{y\}_2$, where x, y are $\mathcal{X}$-coordinates such that $b(x, y) = 0$.

7.1 The A_3 Function

The above is particularly noteworthy because the A_2 function does not have this property. It turns out that the only linear combination of A_2 functions for which this rule holds is the so-called A_3 function, an equally-weighted sum of the six A_2 functions associated with pentagonal faces of A_3:

$$f_{A_3} = \sum_{A_2 \subseteq A_3} f_{A_2} \tag{52}$$

(This depends on a sign convention, or equivalently, an orientation of pentagonal faces. I choose A_2 subalgebras to be oriented "outwards", whereas the more usual convention orients them "upwards" and obtains an alternating sum for f_{A_3}.)

7.2 Cluster Functions at Weight 5 and 6

I have attempted to generalize the notion of a "cluster ($\mathcal{X}$-) function" to higher weights than 4.

Suppose, at weight w, we define a "cluster function" recursively to be a function whose coproduct is a linear combination of antisymmetric tensor products of cluster functions in cluster $\mathcal{X}$-coordinates:

$$\delta(f(x_1, \ldots, x_n)) = \sum_i a_i \left(f_{i1}(x_{i1}) \wedge \cdots \wedge f_{ik}(x_{ik}) \right) \tag{53}$$

The a_i are real numbers, f_{ij} are cluster functions of weight $w_{ij} < w$, and x_{ij} are $\mathcal{X}$-coordinates on some cluster algebra. A weight-one cluster function is simply the logarithm of an $\mathcal{X}$-coordinate. For weight ≤ 4, this should coincide with the usual definition of a cluster $\mathcal{X}$-function.

7.2.1 The Coproduct at Higher Weight

On $\mathcal{L}_5$, δ has two components $\delta_{3,2}$ and $\delta_{4,1}$, these being the restrictions to $B_3 \otimes B_2$ and $\mathcal{L}_4 \otimes \mathbb{C}^*$ respectively. Applying δ a second time produces the components:

$$\begin{aligned} \delta_{3,1,1} &:= \delta|_{\mathrm{B}_3 \otimes \Lambda^2 \mathbb{C}^*} \\ \delta_{2,2,1} &:= \delta|_{\Lambda^2 \mathrm{B}_2 \otimes \mathbb{C}^*} \end{aligned} \tag{54}$$

Each arrow in the following diagram represents a component of the coproduct:

$$\begin{tikzcd}
 & \mathcal{L}_5 \arrow[dl] \arrow[dr] & \\
\mathrm{B}_3 \otimes \mathrm{B}_2 \arrow[d] \arrow[drr] & & \mathcal{L}_4 \otimes \mathbb{C}^* \arrow[d] \arrow[dll] \\
\mathrm{B}_3 \otimes \Lambda^2 \mathbb{C}^* & & \Lambda^2 \mathrm{B}_2 \otimes \mathbb{C}^*
\end{tikzcd}$$

The lower four arrows in the diagram act on the bases of their domains as follows:

$$\begin{aligned}
\delta_{3,1,1}(\{x\}_3 \otimes \{y\}_2) &= \{x\}_3 \bigotimes (1+y) \wedge y \\
\delta_{3,1,1}(\{x\}_4 \otimes y) &= \{x\}_3 \bigotimes x \wedge y \\
\delta_{3,1,1}(f_{A_2}(x_1, x_2) \otimes y) &= 5 \sum_{i=1}^{5} \Big(\{x_{i+1}\}_3 \bigotimes x_i \wedge y - \{x_i\}_3 \bigotimes x_{i+1} \wedge y \Big) \\
\delta_{2,2,1}(\{x\}_3 \otimes \{y\}_2) &= -\{x\}_2 \wedge \{y\}_2 \bigotimes x \\
\delta_{2,2,1}(\{x\}_4 \otimes y) &= 0 \\
\delta_{2,2,1}(f_{A_2}(x_1, x_2) \otimes y) &= \sum_{i,j=1}^{5} j \{x_i\}_2 \wedge \{x_{i+j}\}_2 \bigotimes y
\end{aligned} \tag{55}$$

Consider arbitrary elements $b_{32} \in \mathrm{B}_3 \otimes \mathrm{B}_2$ and $b_{41} \in \mathcal{L}_4 \otimes \mathbb{C}^*$. Then there exists a function $f_5 \in \mathcal{L}_5$ whose coproduct components are b_{32} and b_{41} if and only if $\delta^2 f_5 = 0$, that is, iff the following two integrability conditions are satisfied:

$$\begin{aligned} \delta_{3,1,1}(b_{32}) + \delta_{3,1,1}(b_{41}) = 0 \\ \delta_{2,2,1}(b_{32}) + \delta_{2,2,1}(b_{41}) = 0 \end{aligned} \tag{56}$$

For weight 6, we have the following diagram for δ:

$$\begin{tikzcd}
 & \mathcal{L}_6 \arrow[dl] \arrow[d] \arrow[dr] & \\
\Lambda^2 \mathrm{B}_3 \arrow[dr] & \mathcal{L}_4 \otimes \mathrm{B}_2 \arrow[dl] \arrow[d] \arrow[dr] & \mathcal{L}_5 \otimes \mathbb{C}^* \arrow[dl] \arrow[d] \\
\Lambda^3 \mathrm{B}_2 & \mathrm{B}_3 \otimes \mathrm{B}_2 \otimes \mathbb{C}^* & \mathcal{L}_4 \otimes \Lambda^2 \mathbb{C}^*
\end{tikzcd}$$

The action of δ is too complicated to write explicitly in terms of the basis of each space.

For elements $b_{33} \in \Lambda^2 B_3, b_{42} \in \mathcal{L}_4 \otimes B_2, b_{51} \in \mathcal{L}_5 \otimes \mathbb{C}^*$, the integrability conditions are:

$$\begin{aligned} \delta_{2,2,2}(b_{33}) + \delta_{2,2,2}(b_{42}) + \delta_{2,2,2}(b_{51}) = 0 \\ \delta_{3,2,1}(b_{33}) + \delta_{3,2,1}(b_{42}) + \delta_{3,2,1}(b_{51}) = 0 \\ \delta_{4,1,1}(b_{33}) + \delta_{4,1,1}(b_{42}) + \delta_{4,1,1}(b_{51}) = 0 \end{aligned} \tag{57}$$

7.2.2 Results

There is a weight-five cluster function F_5 with the coproduct:

$$\delta(\mathrm{F}_5(x)) = \{x\}_3 \otimes \{x\}_2 - \{x\}_4 \otimes (1+x) \tag{58}$$

To see that this is a cluster function, note that $\delta^2(\mathrm{F}_5(x)) = 0$ and that $1+x$ factors into $\mathcal{X}$-coordinates for any connected cluster algebra.
At weight six, we have the similar cluster function F_6:

$$\delta(\mathrm{F}_6(x)) = \{x\}_4 \otimes \{x\}_2 - 2\{x\}_5 \otimes (1+x) + \mathrm{F}_5(x) \otimes x \tag{59}$$

Again, it is reasonably straightforward to verify that $\delta^2(\mathrm{F}_6(x)) = 0$.

The symbol for $\mathrm{F}_5(x)$ (for a convenient choice of products of lower-weight functions) is given by:

$$-2\left[(1+x) \otimes x \otimes x \otimes x \otimes (1+x)\right] \tag{60}$$

. That is,

$$\mathrm{F}_5(x) = -2 \int \mathrm{Li}_4(x) \mathrm{dlog}(1+x) \tag{61}$$

Similarly,

$$\begin{aligned} \mathrm{F}_6(x) &= -2\left[(1+x) \otimes x \otimes x \otimes x \otimes (1+x) \otimes x\right] \\ &= \int \mathrm{F}_5(x) \mathrm{dlog}\, x \end{aligned} \tag{62}$$

On A_2 and A_3, I have found computationally that all cluster functions of weight 5 and 6 are linear combinations of Li_5, F_5, Li_6, and F_6.

An interesting feature of these functions is that the function $f_{A_2}(x_1, x_2)$ does not appear in their coproducts. A concerning feature is that F_6 has no $\Lambda^2\mathrm{B}_3$ component in its coproduct. This implies that this definition of a "cluster function" is too restrictive to be physically useful, because it is known that the three-loop six-particle (that is, the weight 6 A_3) amplitude has nonzero $\Lambda^2\mathrm{B}_3$ coproduct.

7.3 Iterated Coproducts at Four Loops

Suppose we have the symbol of a weight 8 function, F. Consider the $\mathcal{L}_6 \otimes B_2$ component of its coproduct:

$$\delta_{6,2}F = \sum_i f_i \otimes g_i \tag{63}$$

Note that f_i is weight 6 and g_i is weight 2. We can take the $\mathcal{L}_4 \otimes B_2$ component of $\delta(f_i)$:

$$\begin{aligned}\delta_{4,2,2}F &= \sum_i \delta_{4,2}(f_i) \otimes g_i \\ &= \sum_i \left(\sum_j f'_j \otimes g'_j \right) \otimes g_i\end{aligned} \tag{64}$$

Taking the $\Lambda^2 B_2$ component of $\delta(f'_j)$ we find the full **iterated coproduct** of F:

$$\begin{aligned}\delta_{2,2,2,2}F &= \sum_i \left(\sum_j \delta_{2,2}(f'_j) \otimes g'_j \right) \otimes g_i \\ &= \sum_i \left(\sum_j \left(\sum_k f''_k \wedge g''_k \right) \otimes g'_j \right) \otimes g_i\end{aligned} \tag{65}$$

All remaining symbols are now of weight 2. This generalizes in the obvious manner to all even weights – that is, to all integer loop orders.

We are interested in the iterated coproduct of $R_6^{(L)}$, the L-loop 6-point MHV amplitude in $\mathcal{N} = 4$ SYM. This is previously known for $L = 2, 3$:

$$\begin{aligned}\delta_{2,2}R_6^{(2)} &= 0 \\ \delta_{2,2,2}R_6^{(3)} &= -\frac{1}{8}\left(\{v_1\}_2 \wedge \{x_2^-\}_2\right) \otimes \{v_1\}_2 + \text{dihedral}\end{aligned} \tag{66}$$

My main result is that, at four loops, the iterated coproduct of the 6-point MHV amplitude, $R' := \delta_{2,2,2,2}R_6^{(4)}$, is given by the following 27-term expression:

$$
\begin{aligned}
&\delta_{2,2,2,2}R_6^{(4)} = \\
&-\frac{1}{8}\left(\{e_1\}_2\wedge\{e_3\}_2\right)\otimes\{v_3\}_2\otimes\{x_2^+\}_2-\frac{1}{8}\left(\{e_1\}_2\wedge\{e_3\}_2\right)\otimes\{x_2^+\}_2\otimes\{v_3\}_2+\frac{1}{8}\left(\{v_1\}_2\wedge\{e_1\}_2\right)\otimes\{v_1\}_2\otimes\{v_1\}_2\\
&-\frac{1}{16}\left(\{v_1\}_2\wedge\{v_2\}_2\right)\otimes\{v_1\}_2\otimes\{v_2\}_2-\frac{1}{8}\left(\{v_1\}_2\wedge\{v_2\}_2\right)\otimes\{v_1\}_2\otimes\{x_1^+\}_2+\frac{1}{16}\left(\{v_1\}_2\wedge\{v_2\}_2\right)\otimes\{v_3\}_2\otimes\{v_1\}_2\\
&-\frac{1}{16}\left(\{v_1\}_2\wedge\{x_1^+\}_2\right)\otimes\{v_1\}_2\otimes\{x_1^-\}_2-\frac{1}{8}\left(\{v_1\}_2\wedge\{x_1^+\}_2\right)\otimes\{v_1\}_2\otimes\{x_1^+\}_2+\frac{1}{8}\left(\{v_1\}_2\wedge\{x_1^+\}_2\right)\otimes\{v_2\}_2\otimes\{v_1\}_2\\
&-\frac{1}{16}\left(\{v_1\}_2\wedge\{x_1^+\}_2\right)\otimes\{x_1^+\}_2\otimes\{v_1\}_2+\frac{1}{8}\left(\{v_1\}_2\wedge\{x_2^+\}_2\right)\otimes\{e_1\}_2\otimes\{x_2^-\}_2+\frac{1}{8}\left(\{v_1\}_2\wedge\{x_2^+\}_2\right)\otimes\{e_3\}_2\otimes\{x_1^+\}_2\\
&-\frac{1}{8}\left(\{v_1\}_2\wedge\{x_2^+\}_2\right)\otimes\{e_5\}_2\otimes\{x_3^-\}_2-\frac{1}{8}\left(\{v_1\}_2\wedge\{x_2^+\}_2\right)\otimes\{e_5\}_2\otimes\{x_3^+\}_2+\frac{1}{8}\left(\{v_1\}_2\wedge\{x_2^+\}_2\right)\otimes\{v_1\}_2\otimes\{x_2^-\}_2\\
&-\frac{1}{8}\left(\{v_1\}_2\wedge\{x_2^+\}_2\right)\otimes\{v_1\}_2\otimes\{x_2^+\}_2+\frac{1}{8}\left(\{v_1\}_2\wedge\{x_2^+\}_2\right)\otimes\{v_2\}_2\otimes\{v_2\}_2-\frac{1}{4}\left(\{v_1\}_2\wedge\{x_2^+\}_2\right)\otimes\{x_2^+\}_2\otimes\{v_1\}_2\\
&-\frac{1}{8}\left(\{v_1\}_2\wedge\{x_2^+\}_2\right)\otimes\{x_3^+\}_2\otimes\{v_1\}_2+\frac{1}{8}\left(\{x_1^+\}_2\wedge\{e_1\}_2\right)\otimes\{v_3\}_2\otimes\{x_2^-\}_2+\frac{1}{8}\left(\{x_1^+\}_2\wedge\{e_3\}_2\right)\otimes\{x_1^-\}_2\otimes\{v_2\}_2\\
&-\frac{1}{8}\left(\{x_1^+\}_2\wedge\{e_3\}_2\right)\otimes\{x_2^+\}_2\otimes\{v_1\}_2+\frac{1}{8}\left(\{x_1^+\}_2\wedge\{e_3\}_2\right)\otimes\{x_3^+\}_2\otimes\{v_2\}_2+\frac{1}{8}\left(\{x_1^+\}_2\wedge\{e_4\}_2\right)\otimes\{v_2\}_2\otimes\{x_2^+\}_2\\
&-\frac{1}{8}\left(\{x_1^+\}_2\wedge\{e_4\}_2\right)\otimes\{v_3\}_2\otimes\{v_3\}_2+\frac{1}{8}\left(\{x_1^+\}_2\wedge\{e_4\}_2\right)\otimes\{x_2^+\}_2\otimes\{v_2\}_2+\frac{1}{8}\left(\{x_1^+\}_2\wedge\{x_2^-\}_2\right)\otimes\{x_1^+\}_2\otimes\{v_3\}_2\\
&+\text{dihedral}
\end{aligned}
\tag{67}
$$

7.3.1 Bases

The 27 terms above are derived from a 46,551-term expression in the (overcomplete) basis $(t_1 \wedge t_2) \otimes t_3 \otimes t_4$, with weight-2 symbols of the form:

$$
\begin{aligned}
t_i &= (y_1 \wedge y_2)\\
y_i &\in \{u_i, 1-u_i, y_{u_i}\}\\
u_i &\in \{u, v, w\}
\end{aligned}
\tag{68}
$$

The y_i are products of $\mathcal{X}$-coordinates of A_3, defined in [8].

The first step of my analysis was to transform R' into a basis of cluster functions:

$$
\begin{aligned}
t_i' &= \{x_i\}_2\\
x_i &\in \{e_k, v_k, x_\ell^+, x_\ell^- : 1 \leq k \leq 6, 1 \leq \ell \leq 3\}
\end{aligned}
\tag{69}
$$

The definitions of the $\mathcal{X}$-coordinates x_i can be found in [10].

This basis is far smaller, and this transformation alone is sufficient to shrink our expression by a factor of 36, to 1,291 terms. The change-of-basis transformation is somewhat complicated by the fact that both the dilogarithmic x basis and (to a greater extent) the y basis are overcomplete.

There are 15 dilogarithms $\{x_i\}_2$ and $\binom{9}{2} = 36$ symbols $(y_1 \wedge y_2)$, but only 33 of the latter appear in R. Linear relations between these functions reduce the dimensionality of each space to 10. (In the case of the x basis, these are the five independent Abel relations on A_3.) A transformation M was previously known for the reverse direction, from the 15-dimensional space X of dilogarithms (ignoring Abel relations) to the 33-dimensional space Y. This is obviously nonunique, but its restriction $M' : (X/A) \to Y$ has the correct image (Y/B), where A and B are the subspaces that are zero under the relations within each basis. This yields an isomorphism $M'' : (X/A) \to (Y/B)$.

I obtained an explicit matrix for M' by expressing five $\{x_i\}_2$ in the linearly-independent basis formed by the other ten. To invert M', I extended it arbitrarily to an invertible map $[(X/A) \oplus Z] \to Y$, where Z is an arbitrary 23-dimensional space. (Specifically, I added dummy basis elements $\{z_1\}_2$ through $\{z_{23}\}_2$.) The inverse map $Y \to [(X/A) \oplus Z]$, when restricted to (Y/B), has a nontrivial image only in the (X/A) component.

The remainder of my work consisted of applying the dihedral transformations and Abel relations in clever ways, to reduce the number of terms in my expression.

7.3.2 Dihedral invariance

The 1291-term expression found above is extremely nonunique. One quick way to shorten it is to note that our expression is invariant under the following maps (generators of the dihedral group D_{12}):

$$\begin{aligned}\{v_i, x_i^+, x_i^-, e_i\} \mapsto& \{v_{i+1}, x_{i+1}^-, x_{i+1}^+, e_{i+1}\} \\ \{v_i, x_i^+, x_i^-, e_i\} \mapsto& \{v_{4-i}, x_{4-i}^+, x_{4-i}^-, 1/e_{5-i}\}\end{aligned} \tag{70}$$

(Where subscripts are taken modulo 6 for e and modulo 3 for v, x.)

Therefore, we can divide the entire expression by 12 and write "+dihedral". This is shorthand,

$$x + \text{dihedral} := \sum_{\sigma \in D_{12}} \sigma(x) \tag{71}$$

This doesn't seem useful at first glance, but it allows us to then replace any term with its image under any dihedral map. This gives us a large set of expressions that can be added to our expression for R' to obtain potentially shorter expressions. For instance, the first map gives us

$$(\{e_1\}_2 \wedge \{e_3\}_2) \otimes \{v_3\}_2 \otimes \left\{x_2^+\right\}_2 \mapsto (\{e_2\}_2 \wedge \{e_4\}_2) \otimes \{v_4\}_2 \otimes \left\{x_3^-\right\}_2 \tag{72}$$

and therefore we can add

$$(\{e_1\}_2 \wedge \{e_3\}_2) \otimes \{v_3\}_2 \otimes \left\{x_2^+\right\}_2 - (\{e_2\}_2 \wedge \{e_4\}_2) \otimes \{v_4\}_2 \otimes \left\{x_3^-\right\}_2 \tag{73}$$

without altering the outcome of our dihedral sum. (In other words, this expression is equivalent to zero under a dihedral sum.) In many cases this can shorten our expression by one or two terms.

7.3.3 Abel identities

Another source of potential simplification comes from the six Abel identities on A_3:

$$\begin{aligned}\{v_1\}_2 + \left\{x_3^+\right\}_2 + \left\{x_2^-\right\}_2 + \{e_2\}_2 - \{e_4\}_2 &= 0 \\ \{v_1\}_2 + \left\{x_2^+\right\}_2 + \left\{x_3^-\right\}_2 + \{e_5\}_2 - \{e_1\}_2 &= 0 \\ \{v_2\}_2 + \left\{x_1^+\right\}_2 + \left\{x_3^-\right\}_2 + \{e_6\}_2 - \{e_2\}_2 &= 0 \\ \{v_2\}_2 + \left\{x_3^+\right\}_2 + \left\{x_1^-\right\}_2 + \{e_3\}_2 - \{e_5\}_2 &= 0 \\ \{v_3\}_2 + \left\{x_2^+\right\}_2 + \left\{x_1^-\right\}_2 + \{e_4\}_2 - \{e_6\}_2 &= 0 \\ \{v_3\}_2 + \left\{x_1^+\right\}_2 + \left\{x_2^-\right\}_2 + \{e_1\}_2 - \{e_3\}_2 &= 0\end{aligned} \tag{74}$$

(These have a linear dependence: the alternating sum of all six identities is zero, so we can omit one arbitrarily.)

By multilinearity, we can find expressions like

$$\begin{aligned}&(\{v_1\}_2 \wedge \{e_3\}_2) \otimes \{v_3\}_2 \otimes \left\{x_2^+\right\}_2 \\ +&\left(\left\{x_3^+\right\}_2 \wedge \{e_3\}_2\right) \otimes \{v_3\}_2 \otimes \left\{x_2^+\right\}_2 \\ +&\left(\left\{x_2^-\right\}_2 \wedge \{e_3\}_2\right) \otimes \{v_3\}_2 \otimes \left\{x_2^+\right\}_2 \\ +&(\{e_2\}_2 \wedge \{e_3\}_2) \otimes \{v_3\}_2 \otimes \left\{x_2^+\right\}_2 \\ -&(\{e_4\}_2 \wedge \{e_3\}_2) \otimes \{v_3\}_2 \otimes \left\{x_2^+\right\}_2\end{aligned} \tag{75}$$

which are equal to zero, and can therefore be added to R' to obtain different expressions for it.

7.3.4 Sparsity

Shortening our representation of R' is an instance of a very general mathematical problem: given a vector $\vec{v}$ and a set of vectors $\{\vec{u}_1, \ldots, \vec{u}_N\}$, find the linear combination

$$\vec{w} = \vec{v} + \sum_{i=1}^{N} \alpha_i \vec{u}_i \tag{76}$$

that minimizes the "ℓ_0 norm", or number of nonzero components of $\vec{w}$:

$$||\vec{w}||_0 = |\{i|w_i \neq 0\}| \tag{77}$$

Extremely similar problems are known in the computer science literature, and some are known to be very difficult (NP-hard) in general.

One naïve approach to such a problem is to systematically add the $\vec{u}_i$ vectors, one at a time, attempting to decrease $||\vec{w}||_0$ at each step. I wrote code to carry out this rather tedious incremental task; the outcome from a simple implementation was a 35-term expression, down from 1,291.

There are two sets of $\vec{u}_i$ in this case, for the dihedral and Abel identities respectively. One trick I used heavily was to restrict the terms that can appear using one set, and then add expressions from the other set to attempt to shorten R'. Initially, I used Abel identities to express everything in terms of only 10 of the 15 $\mathcal{X}$-coordinates, and then apply dihedral transformations to the result. It turned out to be much more effective to do the reverse: replace each term with its lexicographically-least dihedral image, and then apply Abel identities (also in lexicographically-least form) systematically.

A more sophisticated algorithm takes advantage of a trick first discovered in the field of compressive sensing. The ℓ_0 norm can be replaced with the ℓ_1 norm,

$$||\vec{w}||_1 = \sum_i |w_i| \tag{78}$$

In theory, it is possible for optimization of the ℓ_1 norm to produce suboptimal results for ℓ_0; however, in this context the two norms behave similarly, and the ℓ_1 norm is far more computationally tractable. In fact, ℓ_1 minimization is a special case of convex optimization, which can be handled by general linear programming techniques (Mathematica has some linear-programming routines built-in). This approach occurred to me after I had developed the above naïve incremental techniques with greatly increased depth and optimization. In the end, the incremental approach brought R' down to 29 terms, and ℓ_1 methods cut it down to 27.

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