Inflation and the Growth of Density Perturbation in the Early Universe

By London Cooper-Troendle
Advised by Prof. Savvas Koushiappas

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Introduction to Cosmology

On astronomical scales, the motion of particles, stars, galaxies, and all of matter is governed primarily by gravity. Gravity is described through general relativity and the Einstein field equations, which relate the presence of mass and energy to the curvature of the universe:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1}$$

In principle these equations can be solved to give the equations of motion for matter in the universe, but the general form of the equations leads to a set of nonlinear differential equations that can only be solved through complicated numerical computations.

Luckily, the observed universe exhibits two important characteristics collectively known as the cosmological principle that help reduce the Einstein equations significantly, and it is through this process that a simple analytic solution can be obtained. The cosmological principle states that on the scale of millions of light years, the universe is seen to be homogeneous and isotropic. This means that large scale structure looks the same from any location (homogeneity) and in any direction (isotropy). Mathematically, our description of the constituent matter in the universe, namely its density and momentum given by the energy-momentum tensor, is invariant under translations and rotations. Furthermore, these symmetries must also be seen in the metric $g_{\mu\nu}$, which describes the geometry of the universe and establishes notions of length, angle, time, etc. The general form of a metric in spherical coordinates that fits the cosmological principle is the Friedmann-Robertson-Walker (FRW) metric $g_{\mu\nu}$ with associated line element ds^2:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + sin^2(\theta)d\phi^2) \right] \tag{2}$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a(t)^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & a(t)^2 r^2 & 0 \\ 0 & 0 & 0 & a(t)^2 r^2 sin^2(\theta) \end{pmatrix} \tag{3}$$

This metric differs from flat spacetime with the inclusion of a scale factor a(t) by which space is stretched, and a constant curvature k equal to -1, 0, or 1 for open, flat, and closed spacetimes respectively. The scale factor arises from the fact that the cosmological principle only finds the universe to be homogeneous across distances and not times, leaving the scale factor a(t) unspecified in general. Homogeneity and isotropy do not rule out a curved spatial surface, but only a constant curvature agrees with the cosmological principle, forcing k to take fixed value.

From this viewpoint, the universe is treated as a roughly evenly distributed collection of galaxies, approximately modeled as a perfect fluid. This allows the energy momentum tensor to be given by

$$T^\mu_\nu = g_{\nu\lambda} T^{\mu\nu} = g_{\nu\lambda}[(\rho + P)u^\mu u^\nu + P g^{\mu\nu}] \tag{4}$$

for density ρ, pressure P, velocity u, and inverse metric $g^{\mu\nu}$, which can be seen to be diagonal in the fluid's rest frame where $u^\mu = (1, 0, 0, 0)$:

$$T^\mu_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{5}$$

This leaves two important equations: one for T_{00}, and one for T_{ij}. Before these equations can be evaluated, the left side of Einstein's equation needs to be expressed in terms of a known quantity: the metric (and its derivatives). The Einstein tensor is defined by the Ricci tensor $R_{\mu\nu}$ and its trace, the scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$, with the equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{uv}R \tag{6}$$

In turn, the Ricci tensor is defined by

$$R_{\mu\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\rho\mu,\nu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\mu} \tag{7}$$

where Γ are the Christoffel symbol symbols given by

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}) \tag{8}$$

with partial derivatives denoted by indices after a comma. From the definition of the Christoffel symbols and that of the FRW metric given by (2), the nonzero Christoffel symbols can be computed directly, giving:

$$\Gamma^{0}_{ij} = g_{ij}a\dot{a} \tag{9}$$

$$\Gamma^{i}_{0j} = \delta_{ij}\frac{\dot{a}}{a} \tag{10}$$

$$\Gamma^{i}_{ij} = \frac{1}{2}g^{ii}g_{ii,j} \tag{11}$$

$$\Gamma^{i}_{jj} = (2\delta_{ij} - 1)\frac{1}{2}g^{ii}g_{jj,i} \tag{12}$$

From this, R_{00} is quickly computed:

$$\begin{aligned} R_{00} &= -\Gamma^{i}_{0i,0} - \Gamma^{i}_{0j}\Gamma^{j}_{0i} \\ &= -3\left[\frac{d}{dt}(\frac{\dot{a}}{a}) + (\frac{\dot{a}}{a})^2\right] \\ &= -3\frac{\ddot{a}}{a} \end{aligned} \tag{13}$$

Before the T_{00} equation can be fully simplified and computed, the Ricci scalar must be found, requiring R_{ij} be computed as well:

$$\begin{aligned} R_{ij} &= \Gamma^{\rho}_{ij,\rho} - \Gamma^{\rho}_{\rho i,j} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{ij} - \Gamma^{\rho}_{j\lambda}\Gamma^{\lambda}_{\rho i} \\ &= g_{ij}\left[\frac{\frac{d}{dt}(a\dot{a})}{a^2} + 3(\frac{\dot{a}}{a})^2 - 2(\frac{\dot{a}}{a})^2 + \frac{2k}{a^2}\right] \\ &= g_{ij}\left[2(\frac{\dot{a}}{a})^2 + (\frac{\ddot{a}}{a}) + \frac{2k}{a^2}\right] \end{aligned} \tag{14}$$

Note that this is the same as in flat space except for the last term containing the curvature. From equations (13) and (14), the scalar tensor R is computed:

$$\begin{aligned} R &= g^{\mu\nu}R_{\mu\nu} \\ &= -R_{00} + g^{ij}g_{ij}\left[2(\frac{\dot{a}}{a})^2 + \frac{\ddot{a}}{a} + \frac{2k}{a^2}\right] \\ &= 6\left[\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{k}{a^2}\right] \end{aligned} \tag{15}$$

With R, R_{00}, and R_{ij} computed, Einstein's equations are finally ready to be computed:

$$\begin{aligned} G_{00} &= R_{00} - \frac{1}{2}g_{00}R \\ &= -3\frac{\ddot{a}}{a} + 3\left[\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{k}{a^2}\right] \\ &= 3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} \\ &= 8\pi G T_{00} = 8\pi G\rho \end{aligned} \tag{16}$$

$$\begin{aligned} G_{ij} &= R_{ij} - \frac{1}{2}g_{ij}R \\ &= g_{ij}\left[2(\frac{\dot{a}}{a})^2 + \frac{\ddot{a}}{a} + \frac{2k}{a^2}\right] - 3\left[\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{k}{a^2}\right] \\ &= -g_{ij}\left[2\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{k}{a^2}\right] \\ &= 8\pi G T_{ij} = 8\pi G g_{ij}P \end{aligned} \tag{17}$$

The general solution given above allows for the possibility of an open or closed universe. However, observational data, including measurements of the Cosmic Microwave Background

(CMB) from the Planck and WMAP satellites, has determined with a roughly three sigma level of certainty that the universe is flat, allowing us to set k=0 and simplify Einstein's equations even further:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \tag{18}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -8\pi G P \tag{19}$$

A linear combination of these equations can be taken to eliminate the $\frac{\dot{a}}{a}$ term, leaving the equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3P + \rho) \tag{20}$$

The recurring quantity $(\dot{a}/a)$, found most notably in equation (18), is called the Hubble parameter $H(t)$, and describes the rate of growth of the scale factor. First observed as an empirical fit to the data showing distant galaxies appearing to move away from us faster based upon how far away they are, the Hubble parameter gives an intuitive sense to the time variation in the scale factor, and currently has a value of roughly $H_0 = 70$ km s^{-1}mpc^{-1}. The analysis so far has not involved expressing quantities in terms of the Hubble parameter, and while it is sufficient to use $(\dot{a}/a)$, thinking in terms of the Hubble parameter can help give a more physical understanding of the larger picture of the evolution of the universe.

Before going further, the pressure and density must be related to the scale factor, which will require an analysis of the energy momentum tensor. To find this relationship, the condition that the covariant derivative vanish is evaluated:

$$T^{\mu}_{\nu;\mu} = \frac{\partial T^{\mu}_{\nu}}{\partial x^{\mu}} + \Gamma^{\mu}_{\lambda\mu} T^{\lambda}_{\nu} - \Gamma^{\lambda}_{\mu\nu} T^{\mu}_{\lambda} = 0 \tag{21}$$

The $\nu = 0$ equation simplifies this, as the only nonzero T^0_μ term is $T^0_0 = -\rho$, yielding an equation that can be solved to give $\rho(a)$ if the relation between pressure and density is specified:

$$\frac{\partial \rho}{\partial t} + \Gamma^{\mu}_{0\mu}\rho + \Gamma^{\lambda}_{0\mu} T^{\mu}_{\lambda} = \frac{\partial \rho}{\partial t} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{22}$$

The first case to consider is non-relativistic matter such as planets, dust and stars, which can be approximated as stationary when compared to the speed of light, setting the pressure to 0. Solving (22) leads to the relation:

$$\rho(a)_{NR} = \frac{\rho_0}{a^3} = \frac{\rho_0}{V} \tag{23}$$

where V is the volume of a hypothetical box of length a. The relation is explained intuitively through the analogy of a gas of fixed number of particles at constant (rest) energy, whose number density and therefore energy density are inversely proportional to the volume of the container.

On the other end of the spectrum is ultra-relativistic matter such as photons and neutrinos, whose velocity is either exactly or nearly the speed of light. These particles travel along null geodesics, and the contraction $g_{\mu\nu}T^{\mu\nu} = T^{\mu}_{\mu} = 3P - \rho = 0$ can be made, leading to an equation of state $P = \frac{1}{3}\rho$ for ultra-relativistic matter. With this pressure-density relation established, $\rho(a)$ can again be computed, this time giving the relation for ultra-relativistic matter:

$$\rho(a)_{UR} = \frac{\rho_0}{a^4} \tag{24}$$

At first this seems surprising, as energy density is not inversely proportional to volume and total energy is not conserved, but this does not actually generate a contradiction. When the scale factor $a(t)$ was introduced into the metric of spacetime, the property of time-invariance, normally taken for granted and used to generate the conserved quantity of energy, was lost. Radiation is simply the first case presented where energy is clearly seen to not be conserved. This is achieved by a stretching of the wavelength of an ultra-relativistic particle as the scale factor grows, leading to an inversely proportional relation between the energy of a particle and the scale factor. Combined with the conservation of total number of particles just as in the non-relativistic case that lead to an a^{-3} proportionality, the total energy density can be seen to vary as a^{-4}.

A third case of interest is for a substance with equation of state $P = -\rho$, given the term cosmological constant after its historical (and coincidental) invention by Einstein, alternatively called dark energy today. In this case the equation for energy density time evolution reduces to $\partial\rho/\partial t = 0$, giving constant energy density even as the universe expands:

$$\rho(a)_\lambda = \rho_0 \tag{25}$$

The properties of a type of matter that acts in this way are not well understood, but its existence in our universe today is strongly supported by measurements of the CMB and of redshifts of distant galaxies, and its gravitation effects can be computed as easily as for normal matter.

With knowledge of $\rho(a)$ for the relevant forms of matter, equation (18) can be solved to give an expression of the time evolution of the scale factor $a(t)$. In a matter dominated non-relativistic universe, setting $\rho(a) = \rho_0/a^3$ leads to a solution of the form:

$$a(t)_{NR} \propto t^{\frac{2}{3}} \tag{26}$$

Similarly, a radiation dominated universe sets $\rho(a) = \rho_0/a^4$, yielding the solution:

$$a(t)_{UR} \propto t^{\frac{1}{2}} \tag{27}$$

Dark energy is the most surprising case, as the constant energy density causes exponential growth (or decay) of the scale factor:

$$a(t)_\lambda \propto e^{H_0 t} \tag{28}$$

The real picture of the universe is more complicated than any of these isolated solutions, as all three types of matter are present. In this case, the total energy density is described by

$$\rho(a) = \rho_{NR} + \rho_{UR} + \rho_\lambda \tag{29}$$

One point to note is that equation (18) holds regardless of the matter composition, and thus the total energy density is fixed - with one exception - by the Hubble parameter. The caveat to this approach is the possibility that the universe is not flat and instead has either positive or negative curvature. The presence of curvature would then serve to balance any perceived discrepancy between the energy density and the Hubble parameter as given in equation (18). Looking back, curvature was omitted on empirical grounds to simplify equations (16) and (17) to (18) and (20), but influenced the value of H similarly to an energy density proportional to $1/a^2$. This leads to the definition of the critical density $\rho_c = 3H^2/8\pi G$, defined to be the value for a given measurement of H such that the universe is indeed flat. Putting all the terms together and expressing energy densities as their current fraction Ω_i of the critical density allows the Hubble parameter to be expressed simply in terms of current values:

$$\frac{H^2}{H_0^2} = \frac{\rho(a)}{\rho_c} = \frac{\Omega_{UR}}{a^4} + \frac{\Omega_{NR}}{a^3} + \frac{\Omega_k}{a^2} + \Omega_\lambda \qquad (30)$$

Since today $H = H_0$ and $a = 1$, a simple relation between each fractional density is created: $1 = \Omega_{UR} + \Omega_{NR} + \Omega_k + \Omega_\lambda$. This allows measuring the values of Ω_{UR}, Ω_{NR}, and Ω_λ from observational data on the CMB as well as the redshift of supernovae and distant galaxies to serve as a probe on the value of Ω_k. This has been done, and the values of the density fractions is as follows: $\Omega_{UR} = 8.24 \times 10^{-5}$, $\Omega_{NR} = 0.27$, $\Omega_\lambda = 0.73$, leaving the curvature of the universe to have a measured value of $\Omega_k = 0.01 \pm .01$.

This seems to paint a nearly complete picture of the current energy density makeup of the universe (leaving out the physical understanding behind the existence of dark energy or its properties), but there is in fact a second mystery - dark matter. Normal, non-relativistic matter is understood very well, and is found to clump heavily into brightly glowing stars; for example, the Sun contains 99.8% of the solar system's mass. This allows a statistical analysis of stars within the Milky Way to determine a mass-luminosity relation that accurately determines the mass of a star of a given brightness, neglecting significantly smaller sources of mass such as planets and dust. Alongside other techniques such as measuring the velocity dispersion, the total (baryonic) mass of distant galaxies can be estimated. Then, sampling a large number of galaxies allows for the density of baryonic matter to be measured, which should agree with the value of Ω_{NR} found previously. However, performing this analysis reveals that baryonic matter is only observed to contribute to 4.4% of the critical density, leaving an unexplained 23 percentage-points of non-relativistic matter, 85% of all non-relativistic matter, dubbed dark matter for its lack of electromagnetic interactions. It is worth noting that dark matter is inferred in other, independent, analysis such as measuring the velocity profile of galaxies from their rotation curve and using equations of motion and gravity to solve for the matter profile, which again differs greatly from measurements of visible matter via mass-luminosity relation. As such, the existence of dark matter is strongly established.

The CMB has been mentioned a few times as an incredibly useful source of observational data on determining cosmological parameters, but its composition and origin have not yet been explained. The CMB is a stream of microwave photons reaching us from every direction. The frequency profile of the CMB is that of a blackbody - because it was emitted from one - characterized by the temperature 2.73K. This immediately begs the question of what omnipresent medium exists in the universe at 2.73K as the source of this radiation, but the answer is a bit unexpected. The source of the CMB photons is a 13.6 billion year old plasma, the baryonic matter present in the early universe. This age corresponds to a redshift of $z = 1100$, meaning that distances between particles were shorter, boosting interaction frequencies compared to what might be expected from the universe today. As an ionized plasma, the hydrogen gas is optically dense to photon radiation, scattering light and reducing the mean free path of photons well below the radius of the universe. At the time of last scattering, the cooling of the universe as it expanded lead to the recombination of electrons with nuclei, de-ionizing the hydrogen gas and allowing photons to travel relatively unimpeded. From this time to today, photons emitted from the ancient hydrogen plasma have traveled on a direct path to their observer, us. As explained with the evolution of radiation density, photons' wavelengths are stretched as the universe expands, redshifting them toward lower and lower frequencies and energies, giving the microwave radiation at 2.73K observed today. Because photons before last scattering interacted frequently with baryonic matter, the two were necessarily in thermodynamic equilibrium, sharing the same temperature. From measuring the CMB temperature today (2.73K) and the redshift of the CMB ($z = 1100 \approx 1/a$), the temperature of photons and thus all baryonic matter at the time of last scattering can be measured to be approximately $4 \times 10^{12} K$.

Because the ionized hydrogen scattered photons, we are unable to look past the time of last scattering to directly observe the universe at earlier times, just as on a cloudy day an observer on the ground cannot look past the clouds that block the view of the sky and stars above. Any analysis of the universe before this time will have to come from more clever, indirect methods. The theory of inflation is the result of such an analysis. When we look at the CMB in different directions, it appears very isotropic, with deviations from the mean temperature of only one part in 10^5. This suggests that these photons coming toward us were previously in thermal equilibrium, for a coincidence of such similar temperatures in every direction is far too improbable. However, CMB radiation has been traveling toward us at the speed of light for 13.6 billion years, and although distances were smaller in the past from a smaller scale factor, our model of a radiation dominated early universe does not allow for the possibility that these distant regions could have made causal contact in the 380,000 years between the big bang and the time of last scattering. Inflation attempts to solve this paradox as well as others through the introduction of a period of exponential growth totaling roughly 60 e-folds in the early universe, well before the time of last scattering.

The picture presented by inflation describes a universe where quickly following the big bang distances between points were incredibly small, and baryonic matter was able to reach thermal equilibrium across all of what will later become the observable universe for us today. Then, at roughly 10^{-36} seconds the energy density of the universe became dominated by the hypothetical inflaton, a scalar field ϕ with associated potential $V(\phi)$. The potential energy of the inflaton far outweighed the kinetic, and the slope of $V(\phi)$ is very shallow so that $V(\phi)$, and therefore the total energy density, is nearly constant. As seen in the case for a universe dominated by dark energy, a constant energy density leads to the exponential growth of the scale factor. As the inflaton slowly slides down the potential well, exponential growth continues until ultimately the scale factor has grown by a factor of e^{60}. At this time, the inflaton has reached the bottom of the potential well, and its energy is mostly kinetic, which it then imparts on the normal matter in the universe. By the end of inflation the inflaton is trapped at the bottom of its potential well with negligible potential and kinetic energy, causing it to no longer significantly influence the growth of the universe, and leaving no direct evidence of its existence.

Before focusing on the specifics of inflation, it is worth discussing some of its other consequences, notably those that solve other former paradoxes of cosmology. First there is the flatness problem, which states that in general the universe need not be as close to perfectly flat as it is (current values place the total energy density within 1% of the critical energy density, less than one standard deviation away from complete agreement and zero curvature). This is an even larger problem in the early universe, where the value of Ω_k is far smaller due to the increase in ρ_{UR}, putting Ω_k as low as 10^{-62} at the Planck era. By increasing the scale factor immensely while maintaining a constant energy density, inflation reduces the relative weight of the k/a^2 term in equation (16) governing expansion. This allows the big bang to create a universe with largely arbitrary energy density compared to the critical density before inflation suppresses the effects of curvature. Then, over the next 13.6 billion years the relative strength of curvature grows as it only varies as $1/a^2$ compared to non- and ultra-relativistic matter which vary with $1/a^3$ and $1/a^4$ respectively, allowing Ω_k to take the current value of $0.01 \pm .01$ today.

One bonus result that inflation gives us for free is the explanation for the origin of density perturbations that eventually grow to form galaxies. Like any field, quantum mechanics dictates that the value of the field is in constant fluctuation. However, unlike other areas where quantum fluctuations do not directly influence the macroscopic world, inflation provides the means for these quantum fluctuations to create small under- and over-densities, which then proceed to gravitationally attract and grow in size over time. These density perturbations are formed when the inflationary expansion of space is uneven because of the fluctuations in ϕ, $V(\phi)$, and therefore H. Expanding space by 60 e-folds then causes these previously microscopic perturbations to be stretched to galactic scales, creating the origin for large scale structure across the universe.

Matter in a Perturbed FRW Metric

To calculate the growth of density perturbations, we first need to formulate equations governing how each type of matter interacts with perturbations in the metric, both in terms of generating perturbations and reacting to them. It is worth noting that the focus of this section is to generate equations of motion for ultra- and non-relativistic matter in the presence of gravity. As such, the section is heavily focused on computation following the description given in Dodelson's Modern Cosmology.

The first area of interest is how ultra-relativistic matter responds to a perturbation, followed by the non-relativistic case. Scalar metric perturbations are considered here because they are coupled to matter, but it is also possible to have tensor and vector perturbations. For scalar perturbations though, the metric is rewritten into the general Cartesian form:

$$ds^2 = -(1 + 2\Psi(x^\mu))dt^2 + a(t)^2(1 + 2\Phi(x^\mu))[dx^2 + dy^2 + dz^2] \tag{31}$$

$$g_{\mu\nu} = \begin{pmatrix} -1 - 2\Psi & 0 & 0 & 0 \\ 0 & a(t)^2(1 + 2\Phi) & 0 & 0 \\ 0 & 0 & a(t)^2(1 + 2\Phi) & 0 \\ 0 & 0 & 0 & a(t)^2(1 + 2\Phi) \end{pmatrix} \tag{32}$$

This tells us that scalar perturbations are defined by two functions: Ψ, the Newtonian gravitational potential, and Φ, the spacial curvature perturbation. However, this definition hides the fact that a choice of gauge has been taken to get this form, particularly Conformal Newtonian gauge. In full generality, scalar metric perturbations can be described by four functions (including Ψ and Φ), but two degrees of freedom allow us to reduce this description to two functions, as detailed in Mukhanov, Feldman, and Brandenberger 1992. Conformal Newtonian gauge is used here due to its readability, as it directly presents the familiar Newtonian gravitational potential Ψ. Keeping this metric in mind, the time evolution of matter perturbations can be computed.

The first problem to tackle is the time evolution of photons, governed by the Boltzmann equation:

$$\frac{df}{dt} = C(f) \tag{33}$$

Starting with the left side, we can express $f = f(t, x, p)$ with the basis of derivatives in the time, position, and momentum that f depends explicitly on

$$\frac{df(t,x,p)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p^i}\frac{dp^i}{dt} \tag{34}$$

or more explicitly in terms of the magnitude of the momentum $p = \sqrt{g_{ij}p^i p^j}$ and the normalized directional momentum $\hat{p}^i = \frac{p^i}{p}$ subject to $\delta_{ij}\hat{p}^i\hat{p}^j = 1$:

$$\frac{df(t,x,p)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} + \frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt} \tag{35}$$

The first simplification is to notice that that both $\partial f/\partial \hat{p}^i$ and $d\hat{p}^i/dt$ are first order perturbations, making the product second order and thus negligible. The zero order distribution $f(t, x, p)$ depends only on the magnitude p and not $\hat{p}^i$, forcing $\partial f/\partial \hat{p}^i$ to be first order. Additionally, $d\hat{p}^i/dt$ measures the change in direction of the photons' velocity, which can only caused by the presence of a gravitational potential as they follow geodesic paths, causing this term to be linear in Ψ and Φ as well. Thus the combined term is a second order contribution and becomes negligible.

Looking at dx^i/dt, we note that $p^\mu \equiv dx^\mu/d\tau$, so that $dx^i/dt = dx^i/d\tau \times d\tau/dx^0 = p^i/p^0$. Invoking the constraint that photons follow a null geodesic and dropping higher order terms of Ψ allows p^0 to be expressed in terms of Ψ and p:

$$p \cdot p = g_{ij}p^ip^j = -(1+2\Psi)(p^0)^2 + (p)^2 = 0 \tag{36}$$

$$p^0 = \frac{p}{\sqrt{1+2\Psi}} = p(1-\Psi) \tag{37}$$

Furthermore, we can express p^i in terms of p and Φ using the definitions of p and $p^i = C\hat{p}^i$:

$$p = \sqrt{g_{ij}p^ip^j} = Ca(t)\sqrt{(1+2\Phi)\delta_{ij}\hat{p}^i\hat{p}^j} = Ca(t)(1+\Phi) \tag{38}$$

$$p^i = C\hat{p}^i = \frac{p(1-\Phi)}{a}\hat{p}^i \tag{39}$$

Notice that in the absence of a gravitational potential, the photon's momentum takes the classical value $p(1, \hat{p}^i)$. Finally, the $\partial f/\partial x^i \times dx^i/dt$ term can be computed, making use of the fact that $\partial f/\partial x^i$ is first order for the same reason $\partial f/\partial p^i$ was, and thus must be multiplied by a zero order term in dx^i/dt to not drop out:

$$\frac{\partial f}{\partial x^i}\frac{dx^i}{dt} = \frac{p^i}{p^0}\frac{\partial f}{\partial x^i} = \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} \tag{40}$$

The next term to consider is dp/dt, which will require analyzing the geodesic equation for photons:

$$\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{d^2x^\lambda}{d\tau^2} = 0 \tag{41}$$

$$\Gamma^0_{\mu\nu}p^\mu p^\nu = -\frac{dp^0}{d\tau} = -\frac{dp^0}{dt}\frac{dt}{d\tau} = -p^0\frac{dp^0}{dt} = -p(1-\Psi)\left[\frac{dp}{dt}(1-\Psi) - p\frac{d\Psi}{dt}\right] \tag{42}$$

Writing $\frac{d\Psi}{dt} = \frac{\partial\Psi}{\partial t} + \frac{\partial\Psi}{\partial x^i}\frac{dx^i}{dt} = \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}$ and rearranging, equation (42) becomes:

$$\frac{dp}{dt} = p\left[\Psi_{,0} + \frac{\hat{p}^i}{a}\Psi_i\right] - \frac{(1+2\Psi)}{p}\Gamma^0_{\mu\nu}p^\mu p^\nu \tag{43}$$

To go further, the Christoffel symbol term must be computed from the definition of the Christoffel symbol:

$$\begin{aligned}\Gamma^0_{\mu\nu}p^\mu p^\nu &= \frac{-1+2\Psi}{2}(g_{0\mu,\nu}+g_{0\nu,\mu}-g_{\mu\nu,0})p^\mu p^\nu \\ &= \frac{-1+2\Psi}{2}(-2g_{00,\mu}p^\mu - 2g_{00,\nu}p^\nu - g_{\mu\nu,0}p^\mu p^\nu)\end{aligned} \tag{44}$$

Expanding the last term in parentheses shows:

$$\begin{aligned}g_{\mu\nu,0}p^\mu p^\nu &= g_{00,0}(p^0)^2 - g_{ij,0}p^i p^j \\ &= -2\Psi_{,0}\,p^2 + a^2[2\Phi_{,0} + 2H(1+2\Phi)]\delta_{ij}p^i p^j \\ &= -2\Psi_{,0}\,p^2 + 2[\Phi_{,0} + H(1+2\Phi)]p^2(1-2\Phi) \\ &= 2p^2[-\Psi_{,0}\,p^2 + \Phi_{,0} + H]\end{aligned} \tag{45}$$

$$\begin{aligned}\Gamma^0_{\mu\nu}\frac{p^\mu p^\nu}{p} &= \frac{-1+2\Psi}{2}[-4\Psi_{,\mu}\,p^\mu + 2p\Psi_{,0} - 2p(\Phi_{,0} + H)] \\ &= p(-1+2\Psi)[-\Psi_{,0} + 2\Psi_{,i}\frac{\hat{p}^i}{a} - \Phi_{,0} - H]\end{aligned} \tag{46}$$

Bringing this term back into equation (43) for dp/dt yields:

$$\frac{dp}{dt} = -p\left(\Phi_{,0} + \frac{\hat{p}^i}{a}\Psi_{,i} + H\right) \tag{47}$$

There is one final term to the left side of the Boltzmann equation that needs calculating, $\partial f/\partial t$. To accomplish this, the photon Bose-Einstein distribution is adjusted for a fractional temperature distribution $\Theta = \delta T/T$:

$$\begin{aligned}f &= \frac{1}{e^{\frac{p}{T(1+\Theta)}} - 1} \\ &\approx f^{(0)} + \frac{\partial f^{(0)}}{\partial T}\delta T \\ &\approx f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta\end{aligned} \tag{48}$$

We are interested in the first order component of the Boltzmann equation, and specifically we can use the first order expansion given above to compute $\partial f/\partial t$:

$$\begin{aligned}
\left.\frac{\partial f}{\partial t}\right|_{f.o.} &= -p\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial p}\Theta\right) \\
&= -p\frac{\partial f^{(0)}}{\partial p}\Theta_{,0} - p\Theta\frac{dT}{dt}\frac{\partial^2 f^{(0)}}{\partial T\partial p} \\
&= -p\frac{\partial f^{(0)}}{\partial p}\Theta_{,0} + p\Theta H\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right) \qquad (49)
\end{aligned}$$

The last equality arises from the fact that for photons $T \propto a$ allowing $\frac{dT}{dt}/T$ to be replaced by H. Finally returning to the Boltzmann equation with all the substitutions we established and omitting zero order terms we get:

$$\begin{aligned}
\left.\frac{df}{dt}\right|_{f.o.} &= -p\frac{\partial f^{(0)}}{\partial p}\Theta_{,0} - p\Theta_{,i}\frac{\hat{p}^i}{a}\frac{\partial f^{(0)}}{\partial p} - p\frac{\partial f^{(0)}}{\partial p}\left(\Psi_{,0} + \frac{\hat{p}^i}{a}\Psi_{,i}\right) \\
&= -p\frac{\partial f^{(0)}}{\partial p}\left(\Theta_{,0} + \frac{\hat{p}^i}{a}\Theta_{,i} + \Phi_{,0} + \frac{\hat{p}^i}{a}\Psi_{,i}\right) \qquad (50)
\end{aligned}$$

With the left hand side of the Boltzmann equation fully calculated, it is time to evaluate the right hand side, $C(f)$. The collision function for photons is primarily determined by the Compton scattering off of electrons via the reaction:

$$e^-(\vec{q}) + \gamma(\vec{p}) \leftrightarrow e^-(\vec{q'}) + \gamma(\vec{p'}) \qquad (51)$$

The photon has energy $E(p) = p$ of course, while the electron's energy can be approximated $E(q) \approx m_e + q^2/2m_e \approx m_e$, with the same done for $E(q')$. The collision term can then be written as:

$$
\begin{aligned}
C(f(\vec{p})) &= \frac{1}{4m_e^2 p}\int \frac{d^3q}{(2\pi)^3}\int \frac{d^3q'}{(2\pi)^3}\int \frac{d^3p'}{2(2\pi)^3 E(p')} \\
&\times M^2 (2\pi)^4 \delta^3(\vec{p}-\vec{p'}+\vec{q}-\vec{q'})\delta\left(p-p'+\frac{q^2}{2m_e}-\frac{q'^2}{2m_e}\right)[f_e(\vec{q'})f(\vec{p'})-f_e(\vec{q})f(\vec{p})] \\
&= \frac{\pi}{4m_e^2 p}\int \frac{f_e(\vec{q})d^3q}{(2\pi)^3}\int \frac{d^3p'}{(2\pi)^3 p'}[f(\vec{p'})-f(\vec{p})] \\
&\times M^2\left(\delta(p-p')+[E_e(q')-E_e(q)]\frac{\partial\delta(p-p'+E_e(q)-E_e(q'))}{\partial E_e(q')}\bigg|_{E_e(q)=E_e(q')}\right) \\
&= \frac{\pi}{4m_e^2 p}\int \frac{f_e(\vec{q})d^3q}{(2\pi)^3}\int \frac{d^3p'}{(2\pi)^3 p'}[f(\vec{p'})-f(\vec{p})] \\
&\times M^2\left(\delta(p-p')+\frac{[(\vec{p}-\vec{p'})\cdot\vec{q}]}{m_e}\frac{\partial\delta(p-p')}{\partial p'}\right) \qquad (52)
\end{aligned}
$$

Here the second equality follows from the first by evaluating the d^3q' integral using the delta function $\delta^3(\vec{p}-\vec{p'}+\vec{q}-\vec{q'})$ and, using the approximations $f_e(\vec{q}+\vec{p}-\vec{p'}) \approx f_e(\vec{q})$ and $E_e(q)-E_e(q') \approx (\vec{p'}-\vec{p})\cdot\vec{q}/m_e$ because the electron momentum is much larger than the photon momentum, and then expands the remaining delta function around this value. Finally, the last equality uses the chain rule to differentiate with respect to photon momentum. Treating $M^2 = 8\pi\sigma_T m_e^2$ as a constant and using the fractional temperature Θ expansion from equation (48), along with the definition of the monopole component of the temperature perturbation, $\Theta_0(\vec{x},t) \equiv \frac{1}{4\pi}\int \Theta(\hat{p}',\vec{x},t)d\Omega'$, helps simplify $C(f)$ even further (and introduces a p'^2 term when switching to $d\Omega$). The first step is to compute the integral of $f_e(\vec{q})d^3q$ which evaluates to n_e and replaces q/m_e with the electron's velocity v_b, and then use symmetry to remove the $\vec{p'}\cdot v_b$ term by arguing that $\int \vec{p'}\cdot\vec{v_b}d\Omega = 0$. This leaves only an integral over the magnitude of

p, leaving the final function for $C(f)$:

$$
\begin{aligned}
C(f(\vec{p})) &= \frac{n_e\sigma_T}{4\pi p}\int \frac{d^3p'}{p'}\left[f^{(0)}(\vec{p'}) - p'\frac{\partial f^{(0)}}{\partial p'}\Theta(\vec{p'}) + f^{(0)}(\vec{p}) - p\frac{\partial f^{(0)}}{\partial p}\Theta(\vec{p})\right] \\
&\times \left(\delta(p-p') + [(\vec{p}-\vec{p'})\cdot\vec{v_b}]\frac{\partial\delta(p-p')}{\partial p'}\right) \\
&= \frac{n_e\sigma_T}{p}\int_0^\infty p'dp' \\
&\times \left[\delta(p-p')\left(-p'\frac{\partial f^{(0)}}{\partial p'}\Theta_0 - p\frac{\partial f^{(0)}}{\partial p}\Theta(\vec{p})\right) + \vec{p}\cdot\vec{v_b}\frac{\partial\delta(p-p')}{\partial p'}(f^{(0)}(\vec{p'}) - f^{(0)}(\vec{p}))\right] \\
&= -pn_e\sigma_T\frac{\partial f^{(0)}}{\partial p}[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v_b}] \qquad (53)
\end{aligned}
$$

With $C(f)$ specified, we can finally create an explicit Boltzmann equation for photons. Equating (50) with (53) yields:

$$\Theta_{,0} + \frac{p^i}{a}\Theta_{,i} + \Phi_{,0} + \frac{p^i}{a}\Psi_{,i} = n_e\sigma_T(\Theta_0 - \Theta + \hat{p}\cdot\vec{v_b}) \qquad (54)$$

There are two final simplifications to be made. First, although equation (54) does describe photon perturbations, it is more useful to express it using the Fourier transformed variable

$$\tilde{\Theta}(\vec{k}) = \int \frac{d^3x}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\Theta(\vec{x}) \qquad (55)$$

where $\partial/\partial x^i$ becomes $i\vec{k}$ with $k^2 = \vec{k}\cdot\vec{k}$. This is because each Fourier mode evolves independently, allowing them to be evaluated one at a time instead of all coupled together. Secondly, the normalized dot product $\vec{k}\cdot\hat{p}/k$ can be expressed in terms of the angle μ between $\vec{k}$ and $\hat{p}$: $cos(\mu) \equiv \vec{k}\cdot\hat{p}/k$, and by assuming that electrons move along the temperature gradient we can set $\vec{v_b}\cdot\hat{p} = \tilde{v}_b\mu$. Rewriting the photon Boltzmann equation then gives:

$$\tilde{\Theta}_{,0} + \frac{ik\mu}{a}\tilde{\Theta} + \tilde{\Phi}_{,0} + \frac{ik\mu}{a}\tilde{\Psi} = n_e\sigma_T(\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v_b}) \qquad (56)$$

The next matter constituent of interest is cold dark matter (CDM). Solving the Boltzmann equation for dark matter will follow the same approach as for photons, namely expanding df/dt

with a basis and computing each component one at a time. The first parallel to be drawn comes from the momentum vector of a dark matter particle, which can be taken almost directly from that of the photon. The primary difference between the two is the nonzero contraction $g_{\mu\nu}p^{\mu}p^{\nu} = -m^2$ for dark matter, leading to an energy $E = \sqrt{-g_{00}p^0p^0} = \sqrt{(p)^2 + m^2}$ (where the definition $p = \sqrt{g_{ij}p^ip^j}$ was kept from the photon case). The analogies to equations (37) and (39) for CDM then become:

$$p^0 = E(1 - \Psi) \tag{57}$$

$$p^i = p\hat{p}^i \frac{1 - \Phi}{a} \tag{58}$$

Since p^0 differs for CDM by a factor of E/p, any term containing p^0, such as $\frac{dx^i}{dt}$, will contain this factor. Alongside the fact that dark matter does not collide, this makes the CDM Boltzmann equation become:

$$\begin{aligned} \frac{df_{DM}}{dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial E}\frac{dE}{dt} + \frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt} \\ &= \frac{\partial f}{\partial t} + \frac{p\hat{p}^i}{aE}\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial E}(H\frac{p}{E} + \frac{p}{E}\Psi_{,0} + \frac{p^i}{a}\Psi, i) = 0 \end{aligned} \tag{59}$$

To reach equations of motion from the Boltzmann equation, we take zeroth and first moments by multiplying by $d^3p/(2\pi)^3$ and $p\hat{p}^i(2\pi)^3Ed^3p$ respectively and integrating. The integrals are then solved using the definitions $n_{DM} = \int d^3pf_{DM}/(2\pi)^3$ and $v^i = \int p\hat{p}^id^3pf_{DM}/[(2\pi)^3En_{DM}]$, with the zeroth here and the first moment below:

$$\begin{aligned} &\frac{\partial}{\partial t}\int \frac{d^3p}{(2\pi)^3}f_{DM} + \frac{\partial}{a\partial x^i}\int \frac{p\hat{p}^id^3p}{(2\pi)^3E}f_{DM} - (H + \Phi_{,0})\int \frac{p^2d^3p}{(2\pi)^3E}\frac{\partial f_{DM}}{\partial E} - \frac{\Psi_{,i}}{a}\int \frac{p\hat{p}^id^3p}{(2\pi)^3}\frac{\partial f_{DM}}{\partial E} \\ &= \frac{\partial n_{DM}}{\partial t} + \frac{\partial(n_{DM}v^i)}{a\partial x^i} + 3(H + \Phi_{,0})n_{DM} = 0 \end{aligned} \tag{60}$$

The last term on the first line was omitted as it is higher order, and the third term was integrated by parts after taking the angular integral over $d\Omega$ by setting $dv = dp\partial f_{DM}/\partial p$. Both of these techniques are used again for the first moment, giving the result:

$$\frac{\partial}{\partial t}\int \frac{p\hat{p}^j d^3p}{(2\pi)^3 E} f_{DM} + \frac{\partial}{a\partial x^i}\int \frac{p^2\hat{p}^i\hat{p}^j d^3p}{(2\pi)^3 E^2} f_{DM} - (H + \Phi_{,0})\int \frac{p^3\hat{p}^j d^3p}{(2\pi)^3 E^2}\frac{\partial f_{DM}}{\partial E} - \frac{\Psi_{,i}}{a}\int \frac{p^2\hat{p}^i\hat{p}^j d^3p}{(2\pi)^3 E}\frac{\partial f_{DM}}{\partial E}$$

$$= \frac{\partial(n_{DM}v^j)}{\partial t} + 4Hn_{DM}v^j + \frac{n_{DM}}{a}\Psi_{,j} = 0 \tag{61}$$

By writing $n_{DM} = n_{DM}^{(0)}(1+\delta)$ the n_{DM} terms can be replaced with fractional density perturbation terms δ. Finally, the two equations generated from the zeroth and first moments are transformed to Fourier space, giving the final equations for dark matter perturbations:

$$\tilde{\delta}_{,0} + \frac{ik}{a}\tilde{v} + 3\tilde{\Phi}_{,0} = 0 \tag{62}$$

$$\tilde{v}_{,0} + H\tilde{v} + \frac{ik}{a}\tilde{\Psi} = 0 \tag{63}$$

Perturbations to the Metric resulting from Matter Over-Densities

With the Boltzmann equations for ultra- and non-relativistic matter calculated, the effect of how particles respond to a metric perturbation has been determined. This leaves the reverse effect to be analyzed, how matter perturbations affect the metric. In the same style as unperturbed FRW cosmology, we begin by computing the Einstein tensor through the definitions of the Ricci tensor, Ricci scalar, and Christoffel symbols, using the perturbed metric as described in equation (32). In Fourier space and neglecting higher order terms, the nonzero Christoffel symbols are:

$$\Gamma^0_{00} = \tilde{\Psi}_{,0} \tag{64}$$

$$\Gamma^0_{i0} = ik_i\tilde{\Psi} \tag{65}$$

$$\Gamma^0_{ij} = \delta_{ij}a[\dot{a}(1 + 2\tilde{\Phi} - 2\tilde{\Psi}) + a\tilde{\Phi}_{,0}] \tag{66}$$

$$\Gamma^i_{00} = \frac{ik_i}{a^2}\tilde{\Psi} \tag{67}$$

$$\Gamma^i_{j0} = \delta_{ij}(H + \tilde{\Psi}_{,0}) \tag{68}$$

$$\Gamma^i_{jk} = i\tilde{\Phi}(\delta_{ij}k_k + \delta_{ik}k_j - \delta_{jk}k_i) \tag{69}$$

Using the same trick as always where higher order terms are omitted to simplify the calculation, the Ricci tensor is calculated:

$$\begin{aligned}
R_{00} &= \Gamma^{\mu}_{00,\mu} - \Gamma^{\mu}_{0\mu,0} + \Gamma^{\mu}_{\mu\nu}\Gamma^{\nu}_{00} - \Gamma^{\mu}_{0\nu}\Gamma^{\nu}_{0\mu} \\
&= \Gamma^{i}_{00,i} - \Gamma^{i}_{0i,0} + \Gamma^{i}_{i\nu}\Gamma^{\nu}_{00} - \Gamma^{i}_{0\nu}\Gamma^{\nu}_{0i} \\
&= \frac{-k^2}{a^2}\tilde{\Psi} + 3\left(H\tilde{\Psi}_{,0} - 2H\tilde{\Phi}_{,0} - \tilde{\Phi}_{,00} - \frac{\ddot{a}}{a}\right) \qquad (70) \\
R_{ij} &= \Gamma^{\mu}_{ij,\mu} - \Gamma^{\mu}_{i\mu,j} + \Gamma^{\mu}_{\mu\nu}\Gamma^{\nu}_{ij} - \Gamma^{\mu}_{j\nu}\Gamma^{\nu}_{i\mu} \\
&= \left(\Gamma^{0}_{ij,0} - \Gamma^{0}_{i0,j} + \Gamma^{0}_{0\nu}\Gamma^{\nu}_{ij} - \Gamma^{0}_{j\nu}\Gamma^{\nu}_{i0}\right) + \left(\Gamma^{k}_{ij,k} - \Gamma^{k}_{ik,j} + \Gamma^{k}_{k\nu}\Gamma^{\nu}_{ij} - \Gamma^{k}_{j\nu}\Gamma^{\nu}_{ik}\right) \\
&= k_i k_j(\tilde{\Phi} + \tilde{\Psi}) + \delta_{ij}\left[k^2\tilde{\Phi} + a^2(\tilde{\Phi}_{,00} + 6H\tilde{\Phi} - \tilde{\Psi}_{,0}) + (1 + 2\tilde{\Phi} - 2\tilde{\Psi})(a\ddot{a} + 2\dot{a}^2)\right] \qquad (71)
\end{aligned}$$

From these terms the Ricci scalar can be computed, but unfortunately it is a complete mess. Keeping in mind that only the $k_i k_j(\tilde{\Phi} + \tilde{\Psi})$ component of R_{ij} depends on the value of i, the scalar can be computed to be:

$$\begin{aligned}
R &= g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{ij}R_{ij} \\
&= (-1 + 2\tilde{\Psi})\left[\frac{-k^2}{a^2}\tilde{\Psi} + 3\left(H\tilde{\Psi}_{,0} - 2H\tilde{\Phi}_{,0} - \tilde{\Phi}_{,00} - \frac{\ddot{a}}{a}\right)\right] \\
&+ \left(\frac{1 - \tilde{\Psi}}{a^2}\right)\left[k_i k_j(\tilde{\Phi} + \tilde{\Psi}) + \delta_{ij}\left[k^2\tilde{\Phi} + a^2(\tilde{\Phi}_{,00} + 6H\tilde{\Phi} - \tilde{\Psi}_{,0}) + (1 + 2\tilde{\Phi} - 2\tilde{\Psi})(a\ddot{a} + 2\dot{a}^2)\right]\right] \\
&= 6(1 - 2\tilde{\Psi})\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right] + \frac{k^2}{a^2}\left(3\tilde{\Phi} + 2\tilde{\Psi}\right)\tilde{\Psi} + 6(\tilde{\Phi}_{,00} + 4\tilde{\Phi}_{,0} - \tilde{\Psi}_0) \qquad (72)
\end{aligned}$$

With these computed, the Einstein tensor can be computed. Because of cancellations in $g_{\mu\nu}$, it is easier to find the values in mixed form. One simplification to be made is that we will only be interested in the first order perturbations terms, as the zeroth order terms both in $G_{\mu\nu}$ and $T_{\mu\nu}$ have already been computed and analyzed, allowing us to subtract them off without losing anything:

$$
\begin{aligned}
G^0_{0,f.o.} &= g^{00}R_{00} - \frac{1}{2}R \\
&= -6\frac{\ddot{a}}{a}\tilde{\Psi} + \frac{k^2}{a^2}\tilde{\Psi} + 3\tilde{\Phi}_{,00} - 3\frac{\dot{a}}{a}(\tilde{\Psi}_{,0} - 2\tilde{\Phi}_{,0}) \\
&+ 6\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right]\tilde{\Psi} - \frac{k^2}{a^2}\tilde{\Psi} - 3\tilde{\Phi}_{00} + 3\frac{\dot{a}}{a}(\tilde{\Psi}_{,0} - 4\tilde{\Phi}_{,0}) - \frac{2k^2}{a^2}\tilde{\Phi} \\
&= 6\left(\frac{\dot{a}}{a}\right)^2\tilde{\Psi} - 6\frac{\dot{a}}{a}\tilde{\Phi}_{,0} - \frac{2k^2}{a^2}\tilde{\Phi}
\end{aligned} \tag{73}
$$

The benefit of finding the mixed form of the Einstein tensor is that T^0_0 is just the total energy density of all the forms of matter. For dark matter and baryons the definition of the fractional density perturbation δ immediately leads to the realization that these elements contribute $\rho\delta$ to the total energy density perturbations. For photons, since the temperature perturbation was used instead of the density perturbation, we need to write out the energy density using the definition of the probability density function and explicitly integrate to find the density perturbation in terms of the temperature perturbation. Writing out the photon distribution using the same expansion in Θ as before quickly reveals the unperturbed energy density integral as the $f^{(0)}$ part. Integrating the linear term over $d\Omega$ gives the monopole term $d\Omega_0$ by definition, and integrating $\frac{\partial f^{(0)}}{\partial p}$ by parts as before gives a coefficient of 4 when differentiating p^4 (after gaining a p^2 from switching to spherical):

$$
\begin{aligned}
T^0_{0,\gamma} &= -2\int \frac{pd^3p}{(2\pi)^3}\left(f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta\right) \\
&= -\rho_\gamma(1 + 4\theta_0)
\end{aligned} \tag{74}
$$

Doing the same for neutrinos (which are definitely ultra-relativistic) of temperature perturbation N and bringing all the matter terms together for the Einstein equation gives:

$$
-3\left(\frac{\dot{a}}{a}\right)^2\tilde{\Psi} + 3\frac{\dot{a}}{a}\tilde{\Phi}_{,0} + \frac{k^2}{a^2}\tilde{\Phi} = 4\pi G(\rho_{DM}\delta_{DM} + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu N_0) \tag{75}
$$

The second equation to consider is that for G^i_j. A neat mathematical trick will allow us to eliminate the trace of G^i_j (and T^i_j), allowing us to focus only on the traceless component. Looking at the definition of G^i_j and our expressions for R and R_{ij}, it is clear that the only non-diagonal term is $\frac{k_i k_j}{a^2}(\tilde{\Psi} + \tilde{\Phi})$, coming from R_{ij}. Therefore when we apply the projection operator $(\hat{k_i}\hat{k^j} - (1/3)\delta^j_i)$ that removes the diagonal component, we need only compute this term:

$$(\hat{k_i}\hat{k^j} - (1/3)\delta^j_i)G^i_j = (\hat{k_i}\hat{k^j} - (1/3)\delta^j_i)(\tilde{\Psi}\tilde{\Phi}) = \frac{2}{3a^2}k^2(\tilde{\Psi} + \tilde{\Phi}) \tag{76}$$

Applying the operator to the energy-momentum tensor yields a factor of $cos^2(\mu) - 1/3$ multiplied to the integral (where μ is the angle between $\hat{k_i}$ and $\hat{k^j}$), which integrates to two thirds of the second Legendre polynomial, leaving the second moment of the matter distribution (which is 0 for non-relativistic matter): $T^i_j = \frac{8}{3}(\rho_\gamma \Theta_2 + \rho_N N_2)$. Equating this to the G^i_j terms yields a second equation describing the gravitational potentials:

$$(\Phi + \Psi) = -\frac{32\pi G a^2}{k^2}(\rho_\gamma \Theta_2 + \rho_N N_2) \tag{77}$$

The Growth of Quantum Perturbations to the Inflationary Field

As described earlier, inflation is a model designed to fix some of the paradoxes found at the heart of classical FRW cosmology, including the horizon problem of causality and thermal equilibrium of the CMB, the flatness problem, and the existence of modern day galaxy sized density perturbations. The mechanism of inflation is a valid solution to general relativity in a FRW universe (as evidenced by dark energy today), but there is no consensus on the physical explanation motivating the existence of this mechanism at the $10^{-36}s$ regime. However, there are proposed models including extensions to the standard model of particle physics or the existence of axions that could potentially lead to an inflationary era. Therefore, it is still worthwhile to analyze the specifics of inflation to make predictions on proposed inflationary models, such as by establishing bounds on energy levels for candidate particles.

The conceptual idea of inflation rapidly expanding the universe, allowing distant locations today to have previously been in causal contact is appealing, but it is important to understand mathematically why this solution works. This motivates the concept of the Hubble radius, defined to be $1/H$, the inverse of the Hubble parameter. Conceptually, the Hubble radius is approximately how far a particle can interact in the time it takes the universe to double. Thus, sections of the universe separated by distances larger than the Hubble radius will not reach thermal equilibrium through causal interactions.

Remembering that H^2 is proportional to ρ from the FRW time-time Einstein equation, we can see that the Hubble radius decreases as ρ grows larger. For matter and radiation dominated eras where ρ is proportional to a^3 or a^4 respectively, looking backwards in time (and thus in scale factor) shows a decreasing Hubble radius. This is essentially a rewording of the horizon problem, by which the Hubble radius shrinks faster than the scale factor (as you run the universe backwards). An even more direct interpretation is to use comoving coordinates, where comoving distances are not scaled by the scale factor and thus two physical locations remain the same comoving distance apart regardless of time. The comoving Hubble radius then becomes $1/aH$, and because H grows faster than a shrinks as the universe is run backwards, the comoving Hubble radius can be seen to shrink, reducing the range of causal connections in the early universe.

Inflation solves this issue by proposing that energy in the early universe was largely stored in the potential energy of a hypothetical inflaton, whose equation of state gives it constant energy density. This prevents ρ from increasing further as time is rewound, and therefore allows the comoving Hubble radius $1/aH$ to expand rapidly as a shrinks by 60 e-folds as you look back to the beginning of inflation. By growing the comoving Hubble radius, regions of space that otherwise would have existed outside of causal contact can immediately be seen to now exist within each other's sphere of influence, allowing thermal equilibrium to be reached.

Historically, inflation was implemented by a scalar field trapped in a local -but not global- potential well, called a false vacuum. Stuck at the local minimum of the potential, the inflaton would maintain near constant potential energy, giving the desired exponential growth of the scale factor. However, the only way for inflation to end was for the inflaton to quantum tunnel across a potential barrier into the global minimum. Since this occurrence is inherently random, there is no way to coordinate the transition of the inflaton across various locations, preventing inflation from ending in a synchronized fashion and creating huge non-homogeneities that are not observed in the universe today.

Instead of using a false vacuum, modern inflation theories instead rely on other models, most famously the slow roll model. This model puts the scalar field in a potential with a very shallow potential gradient, causing the scalar to transition toward the true vacuum without any quantum tunneling barrier present. However, since the gradient is small, the scalar field only slowly moves down the potential, similar to a ball on a gentle slope. Additionally, a slow roll allows the potential energy to be greater than the kinetic, giving a negative pressure necessary for inflation. Mathematically, the scalar field has total energy density kinetic plus potential, and pressure kinetic minus potential:

$$\rho = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + V(\phi) \tag{78}$$

$$P = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - V(\phi) \tag{79}$$

Substituting these into the Einstein equations (18) and (20), the equation of motion for the scalar field is found to be:

$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} + 3\dot{V} = 0 \tag{80}$$

So when does slow roll inflation stop being a slow roll, and thus stop being inflation? To describe and define a period of slow roll inflation, a pair of slow roll parameters are used:

$$\epsilon = \frac{d}{dt}\left(\frac{1}{H}\right) = -\frac{\dot{H}}{aH^2} \tag{81}$$

$$\delta = \frac{1}{H}\frac{\ddot{\phi}}{\dot{\phi}} = -\frac{1}{H\dot{\phi}}(3H\dot{\phi} + \dot{V}) \tag{82}$$

The first parameter ϵ is just the rate of growth of the Hubble radius. Truly time invariant exponential growth would have a constant Hubble radius and set ϵ to zero, but slow roll inflation is an approximate form of exponential growth, and instead inflation is defined as the period when $\epsilon < 1$. Because of the slow roll, δ also takes small value during inflation. In theory an infinite basis of slow roll parameters could be created to completely describe the dynamics of inflation, but because this is a low order approximation in the $(d\phi/dt)^2 << V(\phi)$ regime, higher order parameters do not need to be considered.

The process of computing the power spectrum of density perturbations begins by analyzing quantum perturbations to the inflationary field, and will eventually involve following these perturbations through the radiation and matter dominated eras. To begin, the scalar field is written as a homogeneous zeroth order component $\phi^{(0)}$ plus a perturbation $\delta\phi$. Invoking the conservation law for the energy momentum tensor given by equation (21) rewritten below gives an equation for the power spectrum of quantum fluctuations to the scalar field.

$$\begin{aligned} \tilde{T}^{\mu}_{0;\mu} &= \frac{\partial \tilde{T}^{\mu}_{0}}{\partial x^{\mu}} + \Gamma^{\mu}_{\lambda\mu}\tilde{T}^{\lambda}_{0} - \Gamma^{\lambda}_{\mu 0}\tilde{T}^{\mu}_{\lambda} = 0 \\ &= \tilde{T}^{0}_{0,0} + 3H\delta\tilde{T}^{0}_{0} + ik_i\delta\tilde{T}^{i}_{0} - H\delta\tilde{T}^{i}_{i} \end{aligned} \tag{83}$$

By dropping higher order effects, each of these terms can be found:

$$\delta\tilde{T}^{0}_{0} = -\frac{\dot{\tilde{\phi}}^{(0)}\delta\dot{\tilde{\phi}}}{a^2} - \dot{V}\delta\tilde{\phi} \tag{84}$$

$$\delta\tilde{T}^{i}_{0} = \frac{ik_i}{a^2}\dot{\tilde{\phi}}^{(0)}\delta\tilde{\phi} \tag{85}$$

$$\delta\tilde{T}^{i}_{j} = \delta^{i}_{j}(\dot{\tilde{\phi}}^{(0)}\delta\dot{\tilde{\phi}} - \dot{\tilde{V}}\delta\tilde{\phi}) \tag{86}$$

Putting these all together, a second order differential equation of $\delta\tilde{\phi}$ in conformal time is established. Furthermore, with the variable swap $\tilde{\phi}' = a\tilde{\phi}$ the $d\delta\tilde{\phi}/d\eta$ term can be removed allowing for an easy solution to be generated:

$$\begin{aligned} 0 &= \frac{d^2\delta\tilde{\phi}}{d\eta^2} + 2aH\frac{d\delta\tilde{\phi}}{d\eta} + k^2\delta\tilde{\phi} \\ &= \frac{d^2\delta\tilde{\phi}'}{d\eta^2} + \left(k^2 - \frac{d^2a/dt^2}{a}\right)\delta\tilde{\phi}' \end{aligned} \tag{87}$$

Since we are looking at quantum fluctuations in ϕ, we treat ϕ as a quantum operator and realize that the equation just established is that for a quantum harmonic oscillator. This lets us write a solution in terms of the raising and lowering operators:

$$\delta\hat{\tilde{\phi}}' = v(k,\eta)\hat{a_k} + v^*(k,\eta)\hat{a_k^\dagger} \tag{88}$$

$$v = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 + \frac{1}{ik\eta}\right) \tag{89}$$

Here $k^2 >> \frac{d^2a/dt^2}{a}$ is assumed by looking at equation (20) and noticing that for inflation this goes to zero. With ϕ' and therefore ϕ specified, the variance can be taken using bracket notation and the power spectrum found:

$$\begin{aligned} < \hat{\tilde{\phi}}'(k,\eta)|\hat{\tilde{\phi}}'(k,\eta') > &= \frac{1}{a^2} < \hat{\tilde{\phi}}(k,\eta)|\hat{\tilde{\phi}}(k,\eta') > \\ &= \frac{1}{a^2}|v(k,\eta)|^2(2\pi)^3\delta^3(\vec{k}-\vec{k}') \\ &= \frac{1}{a^2}(2\pi)^3\delta^3(\vec{k}-\vec{k}')P_{\delta\phi}(k) \\ P_{\delta\phi}(k) &\approx \frac{H^2}{2k^3} \end{aligned} \tag{90}$$

From the power spectrum of inflationary perturbations we want to find the power spectrum of metric perturbations at the time they cross outside the horizon and freeze out until re-entry. To do this, it is easiest to first transform gauge to a metric with a time-space component but still a spatially flat slicing:

$$ds^2 = -(1+2A)dt^2 - 2aB_{,i}\,dx^i dt + \delta_{ij}dx^i dx^j \tag{91}$$

By using Bardeen's velocity $v = ikB - ik\dot{\tilde{\phi}}^{(0)}\delta\tilde{\phi}/(\rho + P)a^2$ we can construct a gauge invariant variable ζ which will let us link inflation power spectra with metric power spectra.

$$\begin{aligned}\zeta &= -aHB - \frac{iaH}{k}v \\ &= -\frac{aH}{\dot{\tilde{\phi}}^{(0)}}\delta\tilde{\phi}\end{aligned} \tag{92}$$

This is achieved by recognizing that at the end of inflation ζ takes on the value $-3\tilde{\psi}/2$, allowing us to immediately express ψ in terms of $\delta\tilde{\phi}$ and thus relate the power spectra at the time of horizon crossing (when $k = aH$ for each k mode):

$$\zeta = -\frac{ik_i H\delta T^0_i}{k^2(\rho + P)} - \tilde{\Psi} = -3aH\tilde{\Theta}_1 k - \tilde{\Psi} = -\frac{3}{2}\tilde{\Psi} \tag{93}$$

$$P_\Psi = \frac{4}{9}P_\zeta = \frac{4}{9}\left(\frac{aH}{\dot{\tilde{\phi}}^{(0)}}\right)^2\Bigg|_{k=aH} = \frac{8\pi GH^2}{9\epsilon k^3}\Bigg|_{k=aH} \tag{94}$$

Now we need only to evaluate how the metric perturbation power spectrum evolves once the horizon expands to meet each wave mode. Because each Fourier mode of density perturbations has a different wavelength, it will re-enter the horizon and "unfreeze" at a different time. The transfer function $T(k)$ attempts to express how each mode has evolved based on how long it has had to develop and what value the Hubble parameter took at that era resulting from the makeup of the energy density in the universe. Mathematically, the transfer function is defined to be the ratio of the scalar potential Φ for wavenumber k measured today divided by the primordial value of that scalar potential:

$$T(k) = \frac{\Phi(k, a_{late})}{\Phi_{prim}(k, a_{late})} \tag{95}$$

To fully create the transfer function, the entire set of Einstein equations and Boltzmann equations for each matter constituent must be used in combination to solve for how over-densities grow. Luckily, a number of approximations and simplifications can be made to help reach an analytic solution. First is the realization from equation (77) that in the absence of anisotropies in the form of second moments of the distributions (which we will assume are very small), we can set $\Psi = -\Phi$. Secondly, by considering only either very small or very large wavelength modes we will only have to deal with predominantly one matter constituent at a time: the radiation dominated era or the matter dominated era. Then, once we have equations for both these extremes, we can try to bring them together and solve for the transfer function around the time of radiation-matter equality.

The first case to consider is the regime of large wavelength perturbations, which re-enter the horizon at late times and therefore the matter dominated era. Because of the large wavelength, we can make the approximation $k\eta << 1$ and simplify the Boltzmann equations (56) and (62) to the following:

$$\dot{\Theta}_0 = -\dot{\Phi} \tag{96}$$

$$\dot{\delta}/3 = -\dot{\Phi} \tag{97}$$

Combining these equations leads to the simplification that $\Theta_0 = \delta/3$, which can be used to help solve the Einstein equation (75):

$$3H(H\tilde{\Phi} + \dot{\tilde{\Phi}}) = 4\pi G\delta(\rho + \frac{4}{3}\rho_\gamma) \tag{98}$$

Making the substitution $y = a/a_{eq} = \rho_{DM}/\rho_\gamma$ allows this equation to be solved analytically after a tedious amount of algebra and calculus. In the end, an equation for $\Phi(y)$ and thus the transfer function on large scales is established:

$$\frac{\Phi(y)}{\Phi(0)} = \frac{1}{10y^3}[16(\sqrt{1+y} - 1) + 9y^3 + 2y^2 - 8y] \tag{99}$$

Notice that on the largest scales ($y- > \infty$) Φ decreases by a tenth, and decreases further for smaller values of y. However, as the matter era dominates, the potential Φ begins to take constant value. Since matter over-densities are proportional to $a\Phi$, a constant potential means a linearly growing matter perturbation. This agrees with what we would expect, that gravity will cause dense regions of space to grow and continue to attract matter and increase in over-density.

To analyze small scale modes, one trick to help simplify equations is to notice that since the universe is radiation dominated, baryonic and dark matter do not affect the gravitational potentials and can be dropped from those equations. With this in mind, taking a linear combination of equations (75) and (77) allows us to generate:

$$\tilde{\Phi} = \frac{6a^2H^2}{k^2}\left[\tilde{\Theta}_0 + \frac{3aH}{k}\tilde{\Theta}_1\right] \tag{100}$$

In addition to the Einstein equation given above, we will use the two Boltzmann equations for radiation, now with the k terms included since we are dealing with small wavelengths:

$$\dot{\tilde{\Theta}}_0 + k\tilde{\Theta}_1 = -\dot{\tilde{\Phi}} \tag{101}$$

$$3\dot{\tilde{\Theta}}_1 - k\tilde{\Theta}_0 = -k\tilde{\Phi} \tag{102}$$

Using the Einstein equation and the second Boltzmann equation, we are able to reduce the first Boltzmann equation to:

$$\frac{d\tilde{\Phi}}{d\eta} + \frac{1}{\eta}\tilde{\Phi} = -\frac{6}{k\eta^2}\tilde{\Theta}_1 \tag{103}$$

Differentiating with respect to η, substituting out for $\frac{d\tilde{\Theta}_1}{d\eta}$, and using the new variable $u = \tilde{\Phi}\eta$ gives a second order differential equation in u gives a second order differential equation whose solution is the spherical Bessel function:

$$\frac{\Phi}{\Phi_p} = 3\left(\frac{sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3}\right) \tag{104}$$

The factor of η^{-3} indicates that for modes that enter the horizon in the radiation era their potential Φ will decay, agreeing with intuition that ultra-relativistic matter perturbations have too great a pressure to grow larger from gravitational attraction.

In addition to radiation perturbations, matter perturbations will begin to evolve in the radiation era. For them, the governing Boltzmann equations become:

$$\dot{\tilde{\delta}} + ikv = -3\dot{\tilde{\Phi}} \tag{105}$$

$$\dot{v} + aHv = ik\tilde{\Phi} \tag{106}$$

Differentiating the first equation and using both equations to eliminate v and $\dot{v}$ leaves a differential equation in δ:

$$\ddot{\delta} + \frac{1}{\eta}\dot{\delta} = S(k, \eta) \tag{107}$$

However, the source term is composed of Φ and its derivatives, and it was just shown that these decay quickly and thus are small in the radiation era. This lets us take the approximation $S(k, \eta) = 0$ and solve the homogeneous equation in δ, giving logarithmic solutions of the form:

$$\delta(k, \eta) = A\Phi_p ln(Bk\eta) \tag{108}$$

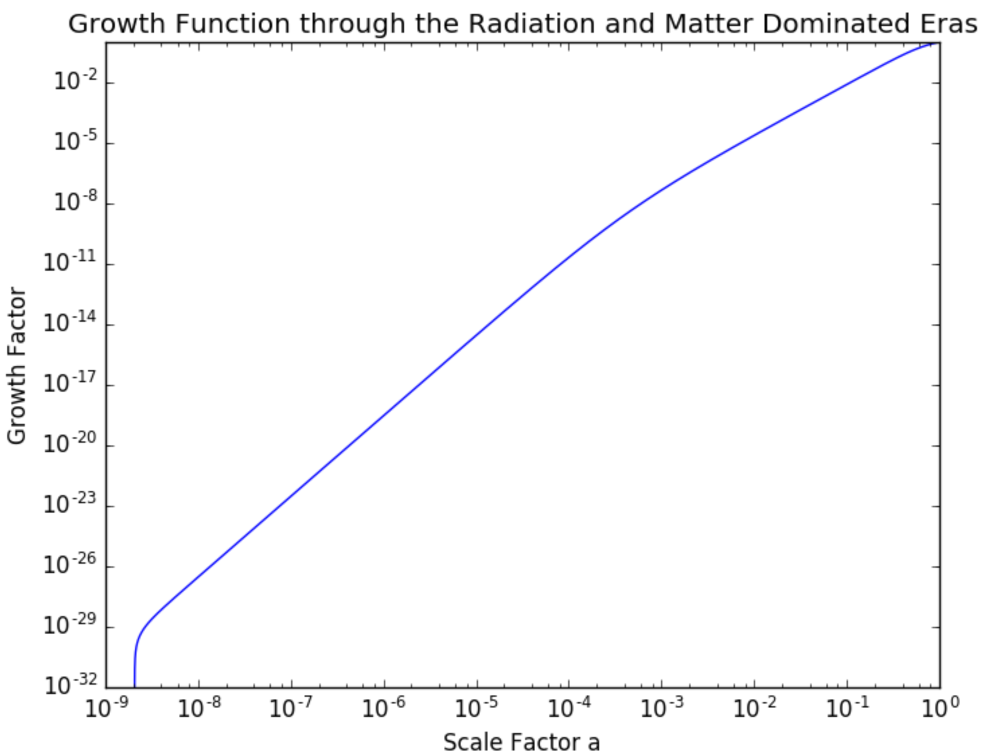
This shows that even in the radiation dominated era matter perturbations still grow, although they are suppressed from linear growth in the matter dominated era to logarithmic growth here.

Eventually, matter perturbations will outgrow radiation perturbations even within the radiation era, because the fractional matter over-density will be large enough compared to the radiation over-density to compensate for a smaller background matter density. At these times, and continuing to the times near or at the era of matter-radiation equality, the approximations we have made so far begin to fall apart. At this point computing the transfer function is much more difficult, and best left to numerical solutions either in full or in an attempt to patch together the early very small and very large wavelength perturbations for which approximate analytic solutions were calculated above.

In contrast to the transfer function, the growth function is concerned with the evolution of perturbations at late times when every mode has crossed into the horizon. This means that different modes are no longer treated differently by an expanding universe because for the scope of the growth function they are all present from the beginning to the present. As a result, the growth function does not depend on wavenumber k, because as long as all the modes have crossed into the horizon they will react the same way to the expansion of the universe. As such, the growth function is set up as an extension of the Meszaros equation at late times and potentially in the presence of new matter elements such as dark energy:

$$D(a) = \frac{5\Omega_m}{2} \frac{H(a)}{H_0} \int_0^a \frac{da'}{(a'H(a')/H_0)^3} \tag{109}$$

The growth factor can be easily calculated numerically by solving for H via the Einstein equations using observed values for the various matter components of the energy density.



Putting together the transfer function and the growth function, the gravitational potential Φ

is defined to evolve through the radiation and matter eras to the present day by the equation:

$$\Phi(\vec{k}, a) = \frac{9}{10}\Phi_p(\vec{k})T(k)\frac{D(a)}{a} \tag{110}$$

The factor of $9/10$ has been factored out of the transfer function so that $T(k) -> 1$ for large wavelengths, and the growth factor has been scaled by a to account for the fact that as defined the growth factor takes value a in a matter dominated regime around $z = 10$ before dark energy becomes relevant.

The Mechanisms of Inflation

Now that density perturbations can be followed from their creation during inflation all the way to $a = 1$ today, it is time to revisit the specific mechanisms of inflation. There are numerous proposed potentials for the scalar field that could drive inflation, including $V = \frac{1}{2}m^2\phi^2$, $V = \lambda\phi^4$, $V = \lambda(\phi^2 - v^2)^2$, $V = \lambda\phi^4 ln(\phi/m_{pl})$, $V = V_0(1 - \lambda\phi^p)$, etc. Here I will investigate the some of the simple inflationary models and their predictions.

The first model to consider is the scalar mass potential $V = \frac{1}{2}m^2\phi^2$. By writing the slow roll parameters as

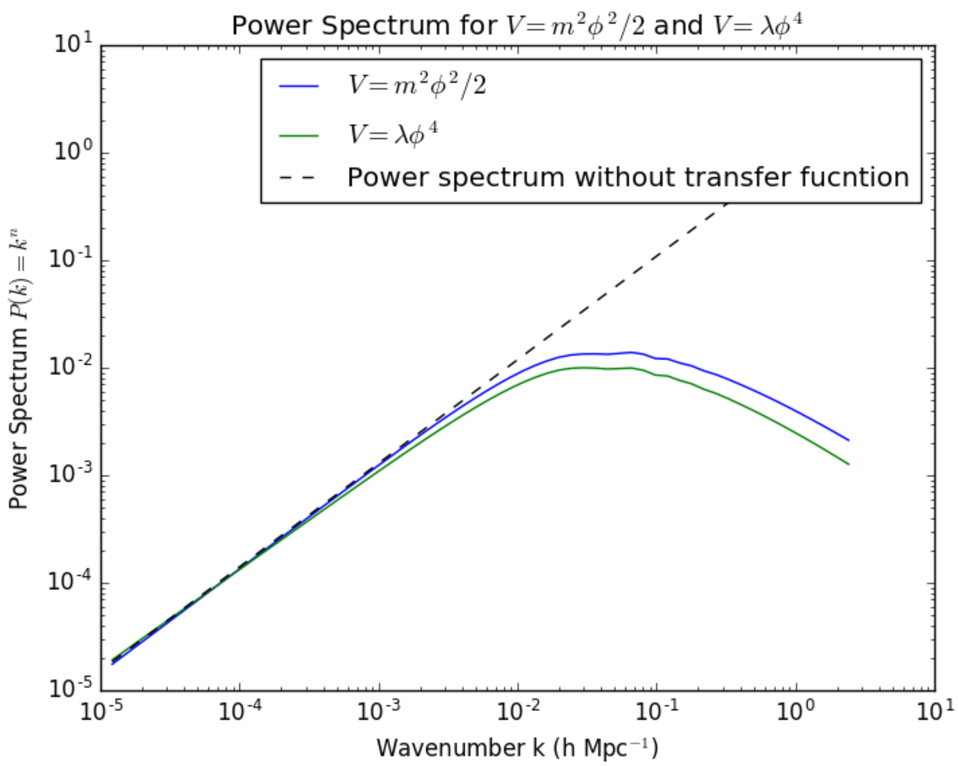
$$\epsilon = \frac{m_{pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \tag{111}$$

$$\delta = \epsilon - \frac{m_{pl}^2}{8\pi}\frac{V''}{V} \tag{112}$$

(where V' denotes a derivative with respect to ϕ) it is easy to understand and compute the slow roll parameters once a potential is established. In this case, $\epsilon = m_{pl}^2/4\pi\phi^2$ and $\delta = 0$. Additionally, the values of ϕ at the start and end of inflation can be calculated once the potential is known. ϕ_ϵ is the point at which inflation ends, defined by when $\epsilon \geq 1$, and takes value $m_{pl}/2\sqrt{\pi}$ for the scalar mass potential. Keeping in mind that roughly 60 e-folds of inflation occur, the starting value of ϕ can also be found by integrating the Hubble parameter, rewritten as:

$$N \approx 60 = \frac{8\pi}{m_{pl}} \int_{\phi_\epsilon}^{\phi_0} \frac{V'}{V} d\phi \tag{113}$$

From this, the starting value of the inflationary field becomes $\phi_0 = \sqrt{31/\pi} m_{pl} \approx 9.869 m_{pl}$. Finally, the spectral index n of the power spectrum $P_k \propto k^n$ can be computed from the previously computed slow roll parameters, using the equation $n = 1 - 4\epsilon - 2\delta$, giving a value of $n = 0.9677$. Data from WMAP places the true value of n at $n = 0.963 \pm 0.012$, which has striking agreement with the value computed for the scalar mass potential given the simplicity and unrefined nature of the computation performed. While the scalar mass potential is capable of generating a spectral index that agrees with observation as seen here, there are other cosmological observations that fall outside the predicted value of parameters established by the model, ruling out its candidacy. An example of a potential that does not generate strong agreement with observation is the simple quartic potential $V = \lambda\phi^4$. From this model it is quickly found that $\epsilon = m_{pl}^2/\pi\phi^2$ and $\delta = -\epsilon/2$. From this, the spectral index is computed to be $n = 0.918$, which certainly falls outside the observational range of the spectral index as listed by WMAP above. The resulting power spectra for both candidate potentials are shown below, alongside the power spectrum in the absence of the transfer function.



In many ways current models of inflation turn into a fine-tuning problem, where a potential designed with appropriately valued parameters aims to agree with all sources of observational data. As the field has evolved, many of the initial candidate potentials have failed to keep up with the increasing precision of observational measurements, ruling out their candidacy as the inflationary potential in the presented format.

To summarize the work presented thus far, we have gone from a general observation on the nature of the universe at large scales, the cosmological principle, and a general description of the motion of matter on these scales, Einstein's general relativity, and proceeded to analyze how matter perturbs our description of the metric, how metric perturbations affect the motion of matter elements, and ultimately used these relations to describe how quantum fluctuations in the inflationary field form density perturbations that grow over time into what are now seen to be galaxies. Toward the end of this analysis, two candidate potentials that could generate slow roll inflation were proposed, and their prediction of the spectral index was computed and compared with the observed value to narrow the viability of these potentials. It was found that the quartic potential did not predict a spectral index in agreement with observations from WMAP, while the quadratic potential did generate a spectral index in agreement. This simple analysis rules out the quartic potential model, and leaves the quadratic model as a possible inflationary potential.

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