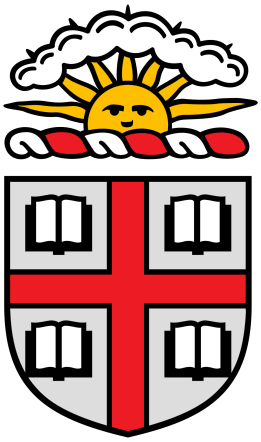
The AdS_3/CFT_2 Correspondence

Entanglement Entropy & Holography



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Abstract

In this study, we explore the correspondence between gravity in anti-de Sitter space (AdS) and quantum fields in conformal field theory (CFT). We first study AdS and CFT individually before proceeding to establish their harmonious relationship. We show that the lengths of spatial geodesics along surfaces of constant proper time in AdS are related to the degree of entanglement entropy harbored in a one-dimensional spatial interval containing a free, massless, scalar field. We discuss the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence in the broader context of the holographic principle, illustrating how physics in a curved D-dimensional spacetime can be described in terms of physics in a corresponding flat $(D-1)$-dimensional spacetime. Along the way, we also present original MATLAB code for making approximate calculations for the entanglement entropy of a free, massless, scalar quantum field in CFT. We show how these calculations produce results which are consistent with the holographic principle.

Contents

Chapter 1: Introduction

Every physical system can be, in principle, described by a comprehensive list of all the information it contains. In a system consisting of classical point-like particles, this information might take the form of a list of the positions and velocities of all the particles at each moment of time, as measured in a certain "laboratory" frame. This information could be summed up in terms of a time-dependent position function for each particle – that is, a trajectory. One can imagine using this method to describe the information contained in systems of arbitrary size, with arbitrary numbers of constituent particles.

If one is lucky, some of the system's particles, which from this point forth we will refer to as *subsystems*, will have trajectories which are correlated to one another. For example, if the system is a completely rigid object, then the motion of every subsystem can be described purely as a vector sum of the linear motion of the object's center of mass and the rotational motion of that subsystem about the center of mass. Thus knowledge of the trajectory of any particular subsystem greatly restricts the set of possible trajectories for the other subsystems. In this case, a complete list of trajectories for every particle would not be necessary to describe all elements of the system; such a list would have a great deal of redundancy. In fact, the only information that would be necessary to deduce the trajectory of a particular subsystem would be the trajectory of the overall system's center of mass, the position of that subsystem relative to the center of mass, and the rotational velocity of the object. Because it takes comparatively so little effort to describe all elements of this system, we say that this system has a low *entropy*.

In undergraduate thermodynamics classes, students generally learn to associate entropy with a system's disorder. While this perspective is not incorrect, it is often useful to associate a system's entropy with the amount of information the system contains – that is, to what degree is it possible to describe a system in terms of patterns and correlations of its constituent subsystems, rather than in terms of a comprehensive list? A system whose subsystems are greatly correlated has a very low entropy, while a system for which there is no relationship between the behaviors of its constituent subsystems is one with maximal

entropy.[1]

Unfortunately, quantifying the entropy of a system is not always a simple task. There are various mathematical definitions for entropy, which depend largely on whether one wishes to treat a system classically or quantum-mechanically.

A natural question to ask is what the maximum amount of entropy – and hence information – is that can be confined in a certain parcel of space. For a d-dimensional parcel of radius R, it is natural to suspect that the maximum entropy is proportional to R^d, scaling with the size of the parcel itself. However, it has been becoming clear to physicists in the past few decades that in actuality, the maximum entropy is proportional to R^{d-1}.[2] This surprising result has come to be known as the *holographic principle*, as it suggests that d-dimensional objects can be described in terms of information encoded in only $(d-1)$-dimensions, just like how a hologram encodes information on a two-dimensional surface to project a three-dimensional image.

The holographic principle provides us with a fascinating new perspective on the nature of the universe, as it suggests that three-dimensional physical objects may themselves be mere projections of information encoded on a two-dimensional surface. In a sense, the holographic principle establishes a link between physical phenomena occurring in two distinct physical spaces whose dimensionality differs by one. A natural question to ask at this point is if this principle can be extended across multi-dimensional spaces as a syllogism: If phenomena in a d-dimensional space can be described in terms of information

[1] Almost paradoxically, some scholars prefer to describe entropy as the *lack* of information or the amount of *hidden* information in a system. This is nothing more than a difference of perspective regarding the precise definition of the word *information*: These scholars are taking the perspective of a *sentient observer*, while this paper is taking the perspective of the *system* itself. From their perspective, a closed system always has the same amount of information but it is the subset of that information which is *hidden* from the sentient observer that determines the entropy. When a system's entropy is low, the observer can describe all the information in the system very easily and hence very little information is hidden; but after the entropy increases substantially (as the Second Law of Thermodynamics dictates it eventually must), the observer will have very little access to the vast amounts of information which the system contains because much of the system's information has become hidden. So from this perspective, higher entropy is associated with the *loss* of information because some of the original accessible information has become hidden. From the author's point of view, there is just "information" – there is no such thing as "hidden information." The system starts out with very little information in the sense that it is very simple to describe; and after a substantial entropy increase, it now harbors *more* information and therefore is more difficult to describe. According to this perspective, the amount of information which the system contains grows over time, and the word "information" could easily be replaced by the word "complexity." What we refer to in this paper as "information" is what others might refer to as "hidden information." We will continue to use language consistent with the author's perspective throughout the course of this paper.

[2] In the special case where $d = 1$, this maximum entropy scales with lnR. So a more general statement would be that, for a d-dimensional parcel with radius R, the maximum entropy scales with $\int R^{d-2} dR$.

encoded in a $(d-1)$-dimensional space, and phenomena in a $(d-1)$-dimensional space can be described in terms of information encoded in a $(d-2)$-dimensional space, then can phenomena in a d-dimensional space be described in terms of information encoded in a $(d-2)$-dimensional space? If so, is it possible to extend this syllogism to make an indefinite chain, thus boiling down phenomena in arbitrary-dimensional spaces to information in a simple 1-dimensional space?

The answer to both these questions is no. What the holographic principle tells us is that it is possible to draw a one-to-one correspondence between the physical states of a d-dimensional gravity-free system and $(d+1)$-dimensional system which includes gravity. Or, expressed in an Einsteinian framework which incorporates the temporal dimension, it is possible to establish a one-to-one correspondence between a curved D-dimensional spacetime and its flat $(D-1)$-dimensional spacetime boundary. It is apparently the presence or absence of curvature – and hence gravity – which determines whether or not the information content harbored within a region of spacetime can be expressed in terms of a lower dimension. This means that the physics of quantum gravity in a given region of spacetime can, in principle, be understood in terms of the gravity-free quantum physics on the lower-dimensional flat boundary region. This is a critical insight as it may help physicists better understand quantum gravity.

A fascinating case study for the holographic principle comes from the AdS/CFT correspondence, which is the main subject of this paper. These are two very different physics models which at first glance have nothing to do with one another and have two completely different parent theories: AdS is a vacuum solution of Einstein's field equations for general relativity, while CFT is a subtheory of quantum field theory. However, a deeper analysis establishes a harmonious connection between these two theories, allowing us to learn about one by studying the other.

Chapter 2: The Geometry of Anti-de Sitter Space

Anti-de Sitter space, often abbreviated AdS, is an exact solution of Einstein's field equations of general relativity characterized by a constant negative spacetime curvature and a vanishing stress-energy tensor (vacuum solution). It is a close cousin of de Sitter space, a vacuum solution with a constant positive spacetime curvature. In general, the cosmological constant Λ in AdS is negative, except in the lowest dimensional case, in which it is zero. This is unlike the actual universe in which the stress-energy tensor does not vanish and in which Λ appears to be small but positive, and so AdS is not an adequate description of our physical universe. However, AdS can still provide insights into the nature of the universe.

In a hypothetical spacetime model, let d denote the number of spatial dimensions and D denote the total number of spacetime dimensions. Typically, there is only one time dimension, so $D = d + 1$, but one can consider models in which this is not the case. In this study, we will be chiefly concerned with the case where $d = 2$ and $D = 3$, but we will try to speak in more general terms where possible. Let us consider a pseudo-Euclidean space of dimension $d + 2$ with metric $g_{\mu\nu} = diag(-1, -1, 1, ..., 1)$. Let us denote the two timelike coordinates u and v and the remaining d spacelike coordinates $x^1, ...x^d$. Now, let us supplement the condition that:

$$-u^2 - v^2 + \sum_{i=1}^{d} (x^i)^2 = -r_{sk}^2 \tag{2.1}$$

Here, r_{sk} is a constant value known as the *skirt radius*. Its value is related to the curvature of the space, with small values corresponding to sharp curvature. In this pseudo-Euclidean space, the spacetime interval ds is given by:

$$ds^2 = -du^2 - dv^2 + \sum_{i=1}^{d} (dx^i)^2 \tag{2.2}$$

In the case where $d = 1$, we could depict this in a three-dimensional Cartesian coordinate system by a hyperboloid. See figure 2.1. Depicting anti-de Sitter space in three spacetime dimensions is more challenging to illustratethe curvature, but we can do our best with a cylinder. See figure 2.2. The boundary of this cylinder, as we will see in later chapters, represents the corresponding CFT space.

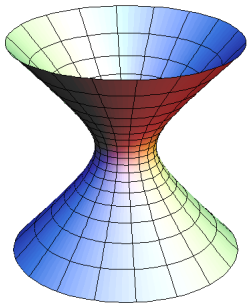

Figure 2.1: A hyperboloid depicting AdS_2. Note the constant negative curvature.

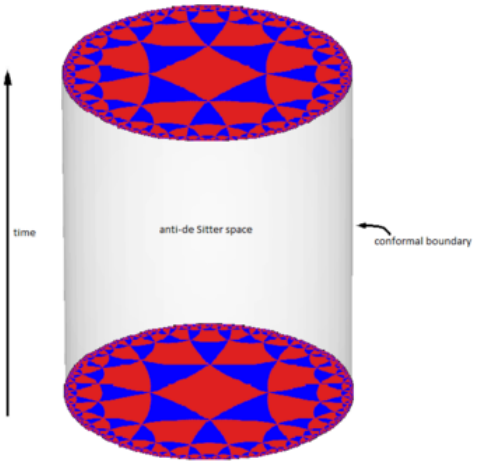

Figure 2.2: A cylindrical depiction of AdS_3. The upward dimension represents time, and the horizontal circular slices (Poincaré disks) represent the two curved spatial dimensions. The conformal boundary represents the corresponding CFT_2. Despite the finite depiction here, AdS_3 has an infinite spatial extent in both spatial dimensions and the temporal dimension.

From this point forth, we will concentrate on the $d = 2$ case ($D = 3$), as this is the lowest dimension in which all the essential material for this study can be captured. We will simply write x for x^1 and y for x^2. That is, our spacetime

distance metric is given by:

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2 \tag{2.3}$$

2.1 Coordinate Representations of AdS

Like many curved geometries, AdS_3 can be coordinatized in a variety of ways. The next few pages detail several of the most common and most useful.

2.1.1 Global Coordinates

We introduce temporal coordinate τ and spatial coordinates ρ and θ such that:[1]

$$u = r_{sk}\, cosh\, \rho\, cos(c\tau) \tag{2.4a}$$
$$v = r_{sk}\, cosh\, \rho\, sin(c\tau) \tag{2.4b}$$
$$x = r_{sk}\, sinh\, \rho\, cos\, \theta \tag{2.4c}$$
$$y = r_{sk}\, sinh\, \rho\, sin\, \theta \tag{2.4d}$$

where c represents the speed of light.[2] τ denotes a proper time coordinate and is not in any way meant to suggest a Euclidean time coordinate.[3] Notice how these coordinates obey the relation $u^2 + v^2 - x^2 - y^2 = 1$, which is the $d = 2$ version of equation (2.1). Using the chain rule to relate u, v, x, τ, and ρ, the spacetime interval is given by:

$$ds^2 = -cosh^2\rho\, d\tau^2 + d\rho^2 + sinh^2\rho\, d\theta^2 \tag{2.5}$$

Here, $\rho \geq 0$ and θ is periodic with period 2π. All the relevant geometric quantities can be calculated and are given as follows:

[1] The observant reader may notice that the arguments of some of the trigonometric functions have units of length, despite the fact that trigonometric functions are always supposed to take unitless arguments. This is because we have implicitly set the skirt radius $r_{sk} = 1$ in the trigonometric arguments, even though we have chosen to include a factor of r_{sk} as a coefficient for each of the coordinates to emphasize the scaling nature of the skirt radius. From this point forth, we will usually assume $r_{sk} = 1$, unless explicitly stated. We will incorporate r_{sk} back into our equations in situations where it provides insight.

[2] For the remainder of this paper, we will set $c = 1$. In general, whenever a temporal coordinate τ or t appears in the equations, it is implicitly multiplied by c so as to produce a quantity with dimensions of length.

[3] The typical time coordinate t is reserved for the Poincaré coordinate formalism, in which t no longer refers to the proper time.

Metric tensor g_{ab} **Inverse metric tensor g^{ab}**

a \ b	τ	ρ	θ	τ	ρ	θ
τ	$-cosh^2\rho$	0	0	$-sech^2\rho$	0	0
ρ	0	1	0	0	1	0
θ	0	0	$sinh^2\rho$	0	0	$csch^2\rho$

Christoffel symbols Γ^a_{bc}

a	τ			ρ			θ		
b \ c	τ	ρ	θ	τ	ρ	θ	τ	ρ	θ
τ	0	$tanh\,\rho$	0	$sinh\,\rho\,cosh\,\rho$	0	0	0	0	0
ρ	$tanh\,\rho$	0	0	0	0	0	0	0	$coth\,\rho$
θ	0	0	0	0	0	$-sinh\,\rho\,cosh\,\rho$	0	$coth\,\rho$	0

Riemann Curvature Tensor R^a_{bcd}

$a = \tau$

b	τ			ρ			θ		
d \ c	τ	ρ	θ	τ	ρ	θ	τ	ρ	θ
τ	0	0	0	0	-1	0	0	0	$-sinh^2\rho$
ρ	0	0	0	1	0	0	0	0	0
θ	0	0	0	0	0	0	$sinh^2\rho$	0	0

$a = \rho$

b	τ			ρ			θ		
d \ c	τ	ρ	θ	τ	ρ	θ	τ	ρ	θ
τ	0	$-cosh^2\rho$	0	0	0	0	0	0	0
ρ	$cosh^2\rho$	0	0	0	0	0	0	0	$-sinh^2\rho$
θ	0	0	0	0	0	0	0	$sinh^2\rho$	0

$a = \theta$

b / c \ d	τ			ρ			θ		
	τ	ρ	θ	τ	ρ	θ	τ	ρ	θ
τ	0	0	$-cosh^2\rho$	0	0	0	0	0	0
ρ	0	0	0	0	0	1	0	0	0
θ	$cosh^2\rho$	0	0	0	-1	0	0	0	0

Ricci tensor R_{ab}

a \ b	τ	ρ	θ
τ	$2cosh^2\rho$	0	0
ρ	0	-2	0
θ	0	0	$-2sinh^2\rho$

Notice that $R_{ab} = -2g_{ab}$. This is a fundamental property of AdS_3.

Ricci Scalar $R = -6$

Note that the Ricci scalar is a constant negative number. In AdS, the curvature does not depend on location or moment within spacetime; this makes sense because the right-hand side of Einstein's field equation also is independent of location or moment within spacetime.[4]

Einstein tensor G_{ab}

a \ b	τ	ρ	θ
τ	$-cosh^2\rho$	0	0
ρ	0	1	0
θ	0	0	$sinh^2\rho$

Notice how $G_{ab} = g_{ab}$. We have demanded that the right-hand side of Einstein's field equation vanishes, so we demand the cosmological constant Λ be set equal to -1.[5]

[4] In fact, the stress-energy tensor is identically equal to 0.

[5] In accurate units, $\Lambda = -1/r_{sk}^2$. Note the negative sign here indicates a negative cosmological constant, indicating a negative vacuum energy, a positive vacuum pressure, and hence a positive contribution to gravitational attraction. This is unlike the actual universe, in which there appears to be a *positive* cosmological constant, indicating a *positive* vacuum energy, a *negative* vacuum pressure, and hence a *negative* contribution to gravitational attraction – that is, gravitational repulsion.

2.1.2 Static Coordinates

An alternate coordinate system can be obtained by introducing coordinate r such that $r = sinh\,\rho$. ($r \geq 0$) This coordinate system is particularly useful for the study of black hole physics in AdS. The resulting metric tensor is given by:[6]

$$ds^2 = -(1+r^2)d\tau^2 + \frac{1}{1+r^2}dr^2 + r^2 d\theta^2 \tag{2.7}$$

A summary of the relevant geometric quantities is as follows:

Metric tensor g_{ab} **Inverse metric tensor g^{ab}**

a \ b	τ	r	θ	τ	r	θ
τ	$-(r^2+1)$	0	0	$-\frac{1}{r^2+1}$	0	0
r	0	$\frac{1}{r^2+1}$	0	0	r^2+1	0
θ	0	0	r^2	0	0	$\frac{1}{r^2}$

Christoffel symbols Γ^a_{bc}

a	τ			r			θ		
c \ b	τ	r	θ	τ	r	θ	τ	r	θ
τ	0	$\frac{r}{r^2+1}$	0	$r(r^2+1)$	0	0	0	0	0
r	$\frac{r}{r^2+1}$	0	0	0	$-\frac{r}{r^2+1}$	0	0	0	$\frac{1}{r}$
θ	0	0	0	0	0	$-r(r^2+1)$	0	$\frac{1}{r}$	0

Riemann Curvature Tensor R^a_{bcd}

$a = \tau$

b	τ			r			θ		
d \ c	τ	r	θ	τ	r	θ	τ	r	θ
τ	0	0	0	0	$-\frac{1}{r^2+1}$	0	0	0	$-r^2$
r	0	0	0	$\frac{1}{r^2+1}$	0	0	0	0	0
θ	0	0	0	0	0	0	r^2	0	0

[6] If one prefers to use units in which the skirt radius r_{sk} is not necessarily of unit length, then we write $r = r_{sk} sinh\,\rho$ and the metric tensor takes the form:

$$ds^2 = -\frac{r_{sk}^2 + r^2}{r_{sk}^2}d\tau^2 + \frac{r_{sk}^2}{r_{sk}^2 + r^2}dr^2 + r^2 d\theta^2 \tag{2.6}$$

However, in our calculations of the geometric quantities for this coordinate system, we will again set $r_{sk} = 1$.

$a = r$

b \ d / c	τ			r			θ		
	τ	r	θ	τ	r	θ	τ	r	θ
τ	0	$-(r^2+1)$	0	0	0	0	0	0	0
r	r^2+1	0	0	0	0	0	0	0	$-r^2$
θ	0	0	0	0	0	0	0	r^2	0

$a = \theta$

b \ d / c	τ			r			θ		
	τ	r	θ	τ	r	θ	τ	r	θ
τ	0	0	$-(r^2+1)$	0	0	0	0	0	0
r	0	0	0	0	0	$\frac{1}{r^2+1}$	0	0	0
θ	r^2+1	0	0	0	$-\frac{1}{r^2+1}$	0	0	0	0

Ricci tensor R_{ab}

a \ b	τ	r	θ
τ	$2(r^2+1)$	0	0
r	0	$-\frac{2}{r^2+1}$	0
θ	0	0	$-2r^2$

Ricci Scalar $R = -6$

Einstein tensor G_{ab}

a \ b	τ	r	θ
τ	$-(r^2+1)$	0	0
r	0	$\frac{1}{r^2+1}$	0
θ	0	0	r^2

2.1.3 Conformal Coordinates

The conformal coordinates can be obtained from the global coordinates by defining a new variable χ such that $r = sinh\,\rho = tan\,\chi$. $(0 \leq \chi < \pi/2)$ The resulting metric is given by:

$$ds^2 = \frac{1}{cos^2\chi}(-d\tau^2 + d\chi^2 + sin^2\chi\, d\theta^2) \tag{2.8}$$

Notice this metric is conformally flat with conformal factor $\frac{1}{cos^2\chi}$. That is, this metric would be flat if not for that scaling factor.

Metric tensor g_{ab} **Inverse metric tensor g^{ab}**

a \ b	τ	χ	θ	τ	χ	θ
τ	$-sec^2\chi$	0	0	$-cos^2\chi$	0	0
χ	0	$sec^2\chi$	0	0	$cos^2\chi$	0
θ	0	0	$tan^2\chi$	0	0	$cot^2\chi$

Christoffel symbols Γ^a_{bc}

a	τ			χ			θ		
b \ c	τ	χ	θ	τ	χ	θ	τ	χ	θ
τ	0	$tan\,\chi$	0	$tan\,\chi$	0	0	0	0	0
χ	$tan\,\chi$	0	0	0	$tan\,\chi$	0	0	0	$\frac{1}{cos\,\chi\,sin\,\chi}$
θ	0	0	0	0	0	$-tan\,\chi$	0	$\frac{1}{cos\,\chi\,sin\,\chi}$	0

Riemann Curvature Tensor R^a_{bcd}

$a = \tau$

b	τ			χ			θ		
d \ c	τ	χ	θ	τ	χ	θ	τ	χ	θ
τ	0	0	0	0	$-sec^2\chi$	0	0	0	$-tan^2\chi$
χ	0	0	0	$sec^2\chi$	0	0	0	0	0
θ	0	0	0	0	0	0	$tan^2\chi$	0	0

$a = \chi$

b	τ			χ			θ		
d \ c	τ	χ	θ	τ	χ	θ	τ	χ	θ
τ	0	$-sec^2\chi$	0	0	0	0	0	0	0
χ	$sec^2\chi$	0	0	0	0	0	0	0	$-tan^2\chi$
θ	0	0	0	0	0	0	0	$tan^2\chi$	0

$a = \theta$

b / c / d	τ			χ			θ		
	τ	χ	θ	τ	χ	θ	τ	χ	θ
τ	0	0	$-sec^2\chi$	0	0	0	0	0	0
χ	0	0	0	0	0	$sec^2\chi$	0	0	0
θ	$sec^2\chi$	0	0	0	$-sec^2\chi$	0	0	0	0

Ricci tensor R_{ab}

a / b	τ	χ	θ
τ	$2sec^2\chi$	0	0
χ	0	$-2sec^2\chi$	0
θ	0	0	$-2tan^2\chi$

Ricci Scalar $R = -6$

Einstein tensor G_{ab}

a / b	τ	χ	θ
τ	$-sec^2\chi$	0	0
χ	0	$sec^2\chi$	0
θ	0	0	$tan^2\chi$

2.1.4 Poincaré Coordinates

The Poincaré coordinates (t, x, z) are related to the conformal coordinates by the following transformations:

$$t = \frac{sin\,\tau}{cos\,\tau + sin\,\chi\,cos\,\theta} \tag{2.9a}$$

$$x = \frac{sin\,\theta\,sin\,\chi}{cos\,\tau + sin\,\chi\,cos\,\theta} \tag{2.9b}$$

$$z = \frac{cos\,\chi}{cos\,\tau + sin\,\chi\,cos\,\theta} \tag{2.9c}$$

For these coordinates, the spacetime interval is given by:

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + dx^2 + dz^2\right) \tag{2.10}$$

If the skirt radius r_{sk} is not already equal to 1, then one must multiply the spacetime interval by r_{sk}^2.

Metric tensor g_{ab} **Inverse metric tensor g^{ab}**

a \ b	t	x	z	t	x	z
t	$-\frac{1}{z^2}$	0	0	$-z^2$	0	0
x	0	$\frac{1}{z^2}$	0	0	z^2	0
z	0	0	$\frac{1}{z^2}$	0	0	z^2

Christoffel symbols Γ^a_{bc}

a	t			x			z		
b \ c	t	x	z	t	x	z	t	x	z
t	0	0	$-\frac{1}{z}$	0	0	0	$-\frac{1}{z}$	0	0
x	0	0	0	0	0	$-\frac{1}{z}$	0	$\frac{1}{z}$	0
z	$-\frac{1}{z}$	0	0	0	$-\frac{1}{z}$	0	0	0	$-\frac{1}{z}$

Riemann Curvature Tensor R^a_{bcd}

$a = t$

b	t			x			z		
d \ c	t	x	z	t	x	z	t	x	z
t	0	0	0	0	$-\frac{1}{z^2}$	0	0	0	$-\frac{1}{z^2}$
x	0	0	0	$\frac{1}{z^2}$	0	0	0	0	0
z	0	0	0	0	0	0	$\frac{1}{z^2}$	0	0

$a = x$

b	t			x			z		
d \ c	t	x	z	t	x	z	t	x	z
t	0	$-\frac{1}{z^2}$	0	0	0	0	0	0	0
x	$\frac{1}{z^2}$	0	0	0	0	0	0	0	$-\frac{1}{z^2}$
z	0	0	0	0	0	0	0	$\frac{1}{z^2}$	0

$a = z$

c \ b	t			x			z		
d	t	x	z	t	x	z	t	x	z
t	0	0	$-\frac{1}{z^2}$	0	0	0	0	0	0
x	0	0	0	0	0	$\frac{1}{z^2}$	0	0	0
z	$\frac{1}{z^2}$	0	0	0	$-\frac{1}{z^2}$	0	0	0	0

Ricci tensor R_{ab}

a \ b	t	x	z
t	$\frac{2}{z^2}$	0	0
x	0	$-\frac{2}{z^2}$	0
z	0	0	$-\frac{2}{z^2}$

Ricci Scalar $R = -6$

Einstein tensor G_{ab}

a \ b	t	x	z
t	$-\frac{1}{z^2}$	0	0
x	0	$\frac{1}{z^2}$	0
z	0	0	$\frac{1}{z^2}$

2.2 The Poincaré Disk

The Poincaré Disk is a two-dimensional subspace of AdS_3 obtained by holding τ constant in the global, static, or conformal coordinate systems, or by setting $t \equiv 0$ in the Poincaré coordinate system. That is, from the perspective of an observer whose temporal coordinate is the proper time τ, all spacetime events on a common Poincaré disk are simultaneous. From (2.5), we see the distance metric in global coordinates is defined by:

$$ds^2 = d\rho^2 + sinh^2\rho\, d\theta^2 \tag{2.11}$$

where $0 \leq \rho < \infty$ and $0 \leq \theta < 2\pi$. Consider a radial segment emanating outwards from the origin ($d\theta = 0$). The length of that segment is given by:

$$s = \int ds = \int d\rho = \rho + cnst \tag{2.12}$$

where the constant is manually set equal to 0. This underscores one of the advantages of using the global coordinate system: one unit in the radial coordinate ρ equals one unit of length. Clearly, this integral will diverge as ρ is permitted to increase without bound. Hence the Poincaré disk represents an infinite two-dimensional space. Nevertheless, it is depicted (as its name would suggest) by a finite disk, though one is supposed to imagine that the disk's boundary is infinitely far away from any point in its interior. These points are called ideal points, omega points, vanishing points, or points at infinity [19].

The global coordinates (ρ, θ) are related to the Euclidean polar coordinates $(r_{Eucl}, \theta_{Eucl})$ by the relation:

$$\rho = ln\left(\frac{1 + r_{Eucl}}{1 - r_{Eucl}}\right) = 2\, tanh^{-1}(r_{Eucl}) \tag{2.13a}$$

$$\theta = \theta_{Eucl} \tag{2.13b}$$

where $0 \leq r_{Eucl} < 1$ and $0 \leq \theta_{Eucl} < 2\pi$ [19].

2.2.1 Geodesics of the Poincaré Disk

Ordinarily in general relativity, the geodesics of principal interest are the timelike and null geodesics, as these are the possible paths which a massive and massless particle could take through spacetime, respectively. However, for purposes of this study, we are primarily interested in a subset of the spacelike geodesics – those with constant proper time, as those paths are confined to the Poincaré disk.

Before solving for the geodesics in terms of the AdS coordinate systems described in the previous subsection, let us first describe the geodesics in a Euclidean framework: the geodesics of the Poincaré disk are given by circular arcs which intersect the disk boundary at a pair of omega points such that a line tangent to the circular arc is perpendicular to a line tangent to the Poincaré disk boundary.[7] Using this simple description, one can derive a general equation for these circular arcs in terms of the Euclidean coordinates r_{Eucl} and θ.[8] The result is:

$$\frac{r_{Eucl}^2 + 1}{2r_{Eucl}}\, cos\,\alpha = cos(\theta - \theta_b) \tag{2.14}$$

Here, θ_b is the angular coordinate of the omega point that lies on the unique radial line bisecting the circular arc, and α is the angular separation between θ_b and either omega point which intersects the arc. We can also depict these

[7] In the special case where these two omega points are directly opposite one another, the geodesic connecting them is simply a straight line running through the center of the disk.

[8] Since $\theta_{Eucl} = \theta$, the angular coordinate for the global, static, and conformal coordinate systems, we can safely drop the subscript on this coordinate.

Figure 2.3: "Angels and Devils" – an artistic rendition of the Poincaré disk by M.C. Escher. One is supposed to imagine that all angels and devils have the same area. The edge of the disk is infinitely far away from all points in the interior.

circular arcs with Euclidean Cartesian coordinates (x, y). In those coordinates, this relation is:

$$\left(x - \frac{cos\,\theta_b}{cos\,\alpha}\right)^2 + \left(y - \frac{sin\,\theta_b}{cos\,\alpha}\right)^2 = tan^2\alpha \tag{2.15}$$

We are now ready to derive the formulas for the geodesics in terms of the coordinate systems described in the previous subsection.[9] Using λ as our affine parameter,[10] we apply the geodesic equations (A.9):

Global coordinates:

$$\begin{cases} \dfrac{d^2\rho}{d\lambda^2} - sinh\,\rho\,cosh\,\rho\left(\dfrac{d\theta}{d\lambda}\right)^2 = 0 & \text{(2.16a)} \\ \dfrac{d^2\theta}{d\lambda^2} + 2\,coth\,\rho\left(\dfrac{d\rho}{d\lambda}\right)\left(\dfrac{d\theta}{d\lambda}\right) = 0 & \text{(2.16b)} \end{cases}$$

$$\text{Solution:} \quad tanh\,\rho\,cos(\theta - \theta_b) = cos\,\alpha \tag{2.16c}$$

[9] Recall that we have imposed the constraint that τ is constant for the global, static, and conformal coordinate systems. For the Poincaré coordinate system, we impose the condition that $t = 0$, as this corresponds to taking $\tau = 0$.

[10] This simply means that λ is some linear function of the distance s: $\lambda = as + b$

Static coordinates:

$$\begin{cases} \dfrac{d^2r}{d\lambda^2} - \dfrac{r}{r^2+1}\left(\dfrac{dr}{d\lambda}\right)^2 - r\left(r^2+1\right)\left(\dfrac{d\theta}{d\lambda}\right)^2 = 0 & \text{(2.17a)} \\ \dfrac{d^2\theta}{d\lambda^2} + \dfrac{2}{r}\left(\dfrac{dr}{d\lambda}\right)\left(\dfrac{d\theta}{d\lambda}\right) = 0 & \text{(2.17b)} \end{cases}$$

$$\text{Solution:} \quad \frac{r}{\sqrt{r^2+1}}\, cos(\theta - \theta_b) = cos\,\alpha \tag{2.17c}$$

Conformal coordinates:

$$\begin{cases} \dfrac{d^2\chi}{d\lambda^2} + tan\,\chi\left[\left(\dfrac{d\chi}{d\lambda}\right)^2 - \left(\dfrac{d\theta}{d\lambda}\right)^2\right] = 0 & \text{(2.18a)} \\ \dfrac{d^2\theta}{d\lambda^2} + \dfrac{2}{cos\,\chi\, sin\,\chi}\left(\dfrac{d\chi}{d\lambda}\right)\left(\dfrac{d\theta}{d\lambda}\right) = 0 & \text{(2.18b)} \end{cases}$$

$$\text{Solution:} \quad sin\,\chi\, cos(\theta - \theta_b) = cos\,\alpha \tag{2.18c}$$

Poincaré coordinates:

$$\begin{cases} \dfrac{d^2x}{d\lambda^2} - \dfrac{2}{z}\left(\dfrac{dx}{d\lambda}\right)\left(\dfrac{dz}{d\lambda}\right) = 0 & \text{(2.19a)} \\ \dfrac{d^2z}{d\lambda^2} + \dfrac{1}{z}\left[\left(\dfrac{dx}{d\lambda}\right)^2 - \left(\dfrac{dz}{d\lambda}\right)^2\right] = 0 & \text{(2.19b)} \end{cases}$$

Solution: Write x and z each as a function of g, which itself is a function of the distance s [20]:[11]

$$\begin{cases} x(g) = \dfrac{l}{2}\, cos\, g & \text{(2.19c)} \\ z(g) = \dfrac{l}{2}\, sin\, g & \text{(2.19d)} \\ g(s) = 2\, tan^{-1}\left(e^s\right) & \text{(2.19e)} \end{cases}$$

The reader is free to verify that these formulas are indeed correct and that they correspond to the Euclidean circular arcs as described in equations (2.14) and (2.15).

[11] l is related to the quantity α by $\alpha = \pi l/L$, where L is the Euclidean circumference of the Poincaré disk.

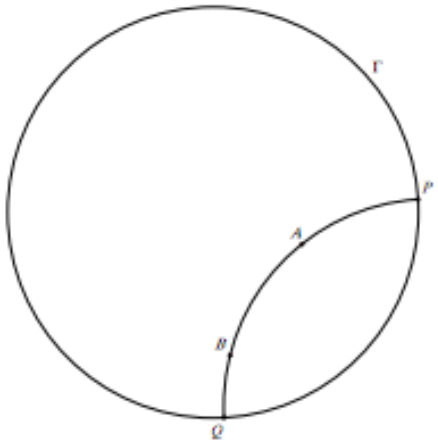

Figure 2.4: A sample geodesic in a Poincaré disk with omega points P and Q and interior points A and B

[19]

2.2.2 Distance and Arc Length in the Poincaré Disk

To find the distance between two interior points A and B, draw the unique geodesic that intersects both of them and extend the geodesic to the boundary. Let the omega points where this geodesic intersects the boundary be denoted P and Q. See figure 2.4.

The distance between A and B is then defined in terms of the Euclidean arc lengths AP, BQ, AQ, and BP as follows [19]:

$$distance(A, B) \equiv \left| ln \left(\frac{AP \cdot BQ}{AQ \cdot BP} \right) \right| \tag{2.20}$$

One can verify that the formula for ρ (2.13b) is valid by applying formula (2.20) to the special case where B is the center of the disk and A is a point at Euclidean radius r_{Eucl} from the center for the disk. One can also verify that any point on the boundary is infinitely far away from any point in the interior:

Theorem 2.2.1. *The distance between any point in the Poincare disk and any ideal point is infinite.*

Proof. Using the distance formula (2.20) for the Poincaré disk, let us calculate the distance between interior point A and ideal point Q by making the

substitution $B = Q$:

$$distance(A, Q) \equiv \left| ln \left(\frac{AP \cdot QQ}{AQ \cdot QP} \right) \right| \tag{2.21}$$

Of course, QQ isn't actually a line segment, but we can think of this as the limiting case in which B approaches Q. Hence the numerator of this fraction is infinitesimal (approaching zero in the limit), while the denominator is positive. Hence the absolute value of the logarithm is increasing without bound in the limit. If both points are ideal ($A = P$), then the numerator is still zero and the denominator is still non-zero, and hence we reach the same result. ■

This theorem verifies that the Poincaré disk has infinite spatial extent. This is important in that it means that length of all geodesics is infinite. As we will see later, in order to speak of the "length" of a geodesic, we must impose a manual cutoff.

Chapter 3: Quantum Information Theory & Quantum Field Theory

The reader who is well-versed in quantum information theory and quantum field theory can feel free to skip to the next chapter; this chapter details key background material in these areas. However, even some advanced readers could benefit from this brief overview.

3.1 Quantum Information Theory

3.1.1 Quantum States & Normalization

A quantum system in an n-dimensional state space has a wave function which can be written:

$$|\psi\rangle = \sum_{k=1}^{n} a_k |a_k\rangle \tag{3.1}$$

where $|a_k\rangle$ represents a basis state with corresponding amplitude a_k. The state must be normalized: $\sum_{k=1}^{n} |a_k|^2 = 1$. Often, it is helpful to choose a basis for a system such that the basis vectors are the eigenvectors of the observable in which one is most interested.

Some quantum observables may require a state space with an infinite number of dimensions, sometimes uncountably infinite. The most typical example is measuring the position of a particle along a one-dimensional axis – there are an uncountably infinite number of possible locations, each of which corresponds to its own unique eigenstate. In this case, it is most efficient to express the wave function as an integral over all possible eigenstates:

$$|\psi\rangle = \int a_k \, da \, |a_k\rangle \tag{3.2}$$

where the upper and lower bounds on the integral correspond to the maximum and minimum values of the observable. The normalization condition is then written as: $\int |a_k|^2 da = 1$.

3.1.2 The Density Operator, Pure States, and Entropy

For every state $|\psi\rangle$, we define a density matrix ρ (often regarded as an operator) as follows:

$$\rho \equiv \frac{|\psi\rangle \langle\psi|}{\langle\psi|\psi\rangle} = \frac{|\psi\rangle \langle\psi|}{Z} \tag{3.3}$$

where $Z \equiv \langle\psi|\psi\rangle$ is the partition function associated with the state $|\psi\rangle$. $|\psi\rangle$ is normalized if and only if $Z = 1$. That is, for any normalized state $|\psi\rangle$, its corresponding density operator ρ satisfies the condition: $tr(\rho) = 1$. For much of this paper (unless otherwise stated), we will assume that $|\psi\rangle$ has already been normalized.

For all quantum states, $tr(\rho^2) \leq 1$ with equality if and only if $|\psi\rangle$ is a *pure state*. We can think of this mathematical condition as being the definition of a pure state, but from an experimentalist's point of view, a pure state is a state for which there exists a quantum observable such that there is a 100% chance of obtaining a certain measured value if one performs a measurement of that observable. Mathematically, this means that there exists a set of basis vectors $\{ |a_k\rangle \}$ spanning the state space such that the amplitudes for all but one of these basis vectors is zero, with the remaining basis vector having an amplitude of magnitude 1.

For a quantum state with density operator ρ, we define the von Neumann entropy S as follows:

$$S(\rho) = -tr(\rho \, log \, \rho) = -\sum_x \lambda_x \, log \, \lambda_x \tag{3.4}$$

where λ_x represents an eigenvalue of the density matrix ρ.[1,2] At this point, it is very natural to ask what the von Neumann entropy of a quantum system in a pure state is. Selecting the eigenvectors $\{ |a_k\rangle \}$ of the relevant observable to be the basis vectors, we can write our state as: $|\psi\rangle = \sum_k \delta_{kj} |a_k\rangle = |a_j\rangle$, where

[1] In the special case where $\lambda_x = 0$, we use the limiting case $\lambda_x \to 0$ instead. So we have $0 \, log \, 0 \equiv 0$.

[2] The base of the logarithm is not of great importance; however, for systems with an n-dimensional state space (for finite n), the base is typically taken to be n. In general, a popular choice for the base is Euler's number e, corresponding to a natural logarithm. For the remainder of this paper, the notations *log* and *ln* will be used somewhat interchangeably, with *ln* being used to emphasize the natural base e in certain cases.

$|a_j\rangle$ is the eigenstate of the system.[3] All but one of the eigenvalues of ρ are then 0, with the lone exception being 1. The von Neumann entropy for a pure state is then:

$$-\sum_{k\neq j}\left[0\,log\,0\right]-1\,log\,1=-0-0=0 \tag{3.5}$$

This is an extremely important result and is in fact a biconditional: *The von Neumann entropy of a quantum state is 0 if and only if the state is pure.*

Theorem 3.1.1. *There is a correspondence between the von Neumann entropy $-tr(\rho\,log\,\rho)$ and the familiar Boltzmann entropy $k_B\,log\,\Omega$ from standard thermodynamics.*

Proof. We consider the case where the vector space is finite-dimensional, and the argument easily generalizes to the infinite-dimensional case, whether countable or not. Assume $|\psi\rangle$ is normalized. We expand out ρ in all its basis states:

$$\rho=|\psi\rangle\langle\psi|=\sum_{k=1}^{n}a_k\,|a_k\rangle\sum_{l=1}^{n}\langle a_l|\,a_l^* \tag{3.6}$$

The (k,l) element of matrix ρ is $a_k a_l$. Taking the classical limit corresponds to eliminate all off-diagonal entries, as this washes out the quantum interference effects:

$$\rho=\sum_{k=1}^{n}\sum_{l=1}^{n}\delta_{k,l}\,a_k\,a_l^*\,|a_k\rangle\langle a_l|=\sum_{k=1}^{n}|a_k|^2\,|a_k\rangle\langle a_k| \tag{3.7}$$

where $|a_k|^2$ now represents the probability p_k of the system being in state $|a_k\rangle$.[4] Thus the eigenvalues of ρ are simply given by $p_k=|a_k|^2$. The von Neumann entropy is then given by:

$$S=-\sum_{k=1}^{n}p_k\,log\,p_k \tag{3.8}$$

This is the Shannon entropy of classical information theory. To make contact with classical thermodynamics, we introduce Ω as the number of microstates corresponding to a particular macrostate. That is, we set: $n=\Omega$. We now impose one of the most fundamental assumptions of statistical mechanics: a system is equally likely to be found in each of its microstates. So for a system in a given macrostate with multiplicity Ω, the probability $p_k=1/\Omega$ for every state. We now have:

$$S=-\sum_{k=1}^{\Omega}\frac{1}{\Omega}\,log\left(\frac{1}{\Omega}\right)=-\Omega\,\frac{1}{\Omega}\,log\,\left(\Omega^{-1}\right)=--log\,\Omega=log\,\Omega \tag{3.9}$$

[3] To be more precise, we could insert a coefficient of the form $e^{i\delta_k}$ for some phase shift δ_k to each basis vector $|a_k\rangle$, but that is not relevant for what follows.

[4] In this case, the lack of certainty is purely classical. ρ is now effectively a one-dimensional distribution function for the probability of the system being in state $|a_k\rangle$.

We now simply insert a factor of Boltzmann's constant k_B and arrive at the famous result:

$$S = k_B \, log \, \Omega \tag{3.10}$$

■

3.1.3 Dividing a System into Subsystems

Now suppose we can partition our system into disjoint subsystems A, B, C, ...[5] Often, we will have exactly two such subsystems, denoted A and B.[6] In this case, we now describe the von Neumann entropy and the density matrix for each subsystem separately by tracing over the basis states associated with the other subsystem:

$$S_A = -tr_A(\rho_A \, log \, \rho_A), \quad \rho_A = tr_B(\rho) \tag{3.11a}$$

$$S_B = -tr_B(\rho_B \, log \, \rho_B), \quad \rho_B = tr_A(\rho) \tag{3.11b}$$

Also note:

$$tr_A(\rho_A) = tr_A[tr_B(\rho)] = tr(\rho) = 1 \tag{3.12a}$$

$$tr_B(\rho_B) = tr_B[tr_A(\rho)] = tr(\rho) = 1 \tag{3.12b}$$

In this study, we will often think of our entire system $A \cup B$ as our entire model universe, with A being the subsystem of interest and B being that subsystem's surroundings. In this case, we now refer to S_A and S_B as the *entanglement entropy* of subsystems A and B, respectively, because this value quantifies the degree of quantum entanglement within the subsystem. The entanglement entropy satisfies three key properties [20]:

1. Entanglement entropy equivalence for complementary subsystems: given two complementary subsystems A and B, $S_A = S_B$.[7] This property is profound in that it allows us to choose whether we wish to trace over a subsystem's density matrix or that of its complement when computing the subsystem's entropy.
2. Subadditivity: For any two disjoint subsystems C and D, $S_C + S_D \geq S_{C \cup D}$.[8]

[5] The subsystems may share a small finite number of boundary points, but their intersection is otherwise empty and their union is the entire system

[6] Some authors write $B = \bar{A}$ to emphasize that A and B are complements of one another.

[7] In this paper, we will only study systems in the idealized case of zero temperature. At finite temperature, this property no longer holds.

[8] This property does not require that $C \cup D$ is the entire system, merely that C and D are disjoint.

3. Strong subadditivity: For any three disjoint subsystems C, D, and E, $S_{C\cup D\cup E} + S_E \leq S_{C\cup E} + S_{D\cup E}$. Notice that in the special case where E is empty, this simply reduces to the regular subadditivity property above.

These properties are stated here without proof.[9] However, a version of property (1) will be proved in Chapter 5 with Srednicki's Holography Theorem. Property (2) is particularly counterintuitive, as one naturally expects entropy to be an extensive quantity and therefore additive: $S_C + S_D = S_{C\cup D}$. However, this is not necessarily the case. The holographic principle teaches us that two disjoint subsystems can contain complementary information, rendering at least some of the information in their union redundant.

We define the family of Rényi entropies as follows:

$$S_A^{(n)} = \frac{1}{1-n} log\left[tr_A\left(\rho_A^n\right)\right] \tag{3.13}$$

with an analogous definition for $S_B^{(n)}$.

Theorem 3.1.2. *The von Neumann entropy is a special limiting case of the Rényi entropy:*

$$S_A = \lim_{n\to 1} S_A^{(n)} \tag{3.14}$$

Proof.

$$S_A^{(n)} = -\frac{ln\left[tr_A\left(\rho_A^n\right)\right]}{n-1} \tag{3.15}$$

Taking the limit as $n \to 1$ requires the use of L'Hôpital's rule:

$$\lim_{n\to 1} S_A^{(n)} = -\lim_{n\to 1} \frac{\partial_n\left(ln\left[tr_A\left(\rho_A^n\right)\right]\right)}{\partial_n(n-1)} = -\lim_{n\to 1} \frac{\partial}{\partial n}\left(ln\left[tr_A\left(\rho_A^n\right)\right]\right) \tag{3.16}$$

This intermediate formula serves another alternate definition for the von Neumann entropy. We continue to simplify the expression:[10]

$$\lim_{n\to 1} S_A^{(n)} = -\lim_{n\to 1} \frac{\partial_n\left[tr_A\left(\rho_A^n\right)\right]}{tr_A\left(\rho_A^n\right)} = -\lim_{n\to 1} \frac{tr_A(\rho_A^n\, ln\, \rho_A)}{tr_A\left(\rho_A^n\right)} = \frac{-tr_A(\rho_A\, ln\, \rho_A)}{tr_A\left(\rho_A\right)} \tag{3.17}$$

The numerator is precisely the entanglement entropy of subsystem A and the denominator is 1 for all density operators:

$$\lim_{n\to 1} S_A^{(n)} = S_A \tag{3.18}$$

■

[9] For proofs and more information on these properties, see Chapter 11 of Nielsen & Chuang's "Quantum Computation and Quantum Information" [16].

[10] The reader can check that $\partial_n\left[tr_A\left(\rho_A^n\right)\right] = tr_A(\rho_A^n\, ln\, \rho_A)$ by expanding out the trace of a matrix as the sum of its eigenvalues and applying property (C.2).

3.2 Quantum Field Theory

The central premise of quantum field theory is that every particle and every wave in the universe can be thought of as an excitation of a quantum field defined over all space and time [13]. Typically, we refer to this quantum field as ϕ, often displaying its dependence on space and time coordinates: $\phi(t, \vec{x})$.[11] ϕ takes the place of the position coordinate x in elementary quantum mechanics. Since $\hat{x}$ is generally thought of as an operator in elementary quantum mechanics, $\hat{\phi}$ must be thought of as an operator in quantum field theory.[12] The field has Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - (\nabla\phi)^2 - m^2\phi^2\right] \tag{3.19}$$

In our study, there is only one spatial dimension and the field is massless. So this reduces to:

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right] \tag{3.20}$$

We define the action S as a functional of ϕ:[13]

$$S[\phi(t,x)] = \int L\,dt = \int\int \mathcal{L}\,dx\,dt = \int\int \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right] dx\,dt \tag{3.21}$$

where L is the Lagrangian, the integral of the Lagrangian density $\mathcal{L}$. We define the conjugate momentum density as:

$$\pi(t,x) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}} \tag{3.22}$$

where $\dot{\phi}$ is a shorthand for $\frac{\partial\phi}{\partial t}$. This conjugate momentum density $\hat{\pi}$ is also an operator and plays a very analogous role in quantum field theory as the momentum operator $\hat{p}$ plays in ordinary quantum mechanics. In this case, $\pi = \dot{\phi}$. We introduce the Hamiltonian density:

$$\mathcal{H} \equiv \pi\dot{\phi} - \mathcal{L} \tag{3.23}$$

The Hamiltonian is given by:

$$H = \int \mathcal{H}\,dx \tag{3.24}$$

[11] In this paper, we will only study quantum fields in $(1+1)$-dimensional spacetimes, so we will drop the vector coordinate and simply write $\phi(t,x)$.

[12] However, for simplicity sake, we will often drop the operator symbol ($\hat{}$), only using it when it is particularly relevant.

[13] The action S is not to be confused with entropy, a quantity also frequently denoted with a capital S. In this paper, the meaning of every S used should be clear from context.

In our case, the Hamiltonian density evaluates to:

$$\mathcal{H} = \frac{1}{2}\left[\dot{\phi}^2 + \left(\frac{\partial\phi}{\partial x}\right)^2\right] \tag{3.25}$$

The operators $\hat{\phi}$ and $\hat{\pi}$ do not commute. Their commutation relation is given by:

$$\left[\hat{\phi}(t,x),\, \hat{\pi}(t',x')\right] = i\hbar\,\delta(t-t')\,\delta(x-x') \tag{3.26}$$

Wave functions $\psi(t,x)$ in quantum mechanics are replaced by wave functionals $\Psi[\phi(t,x)] = \langle\phi(t,x)|\Psi\rangle$ in quantum field theory. These wave functionals must solve the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi[\phi(t,x)] = H\Psi[\phi(t,x)] \tag{3.27}$$

The eigenstates of this equation take the form $|\phi(t,x)\rangle$ for eigenvalue $\phi(t,x)$ and operator $\hat{\phi}(t,x)$:

$$\hat{\phi}(t,x)\,|\phi(t,x)\rangle = \phi(t,x)\,|\phi(t,x)\rangle \tag{3.28}$$

We now have the necessary background material to proceed to study conformal field theory in the next two chapters.

Chapter 4: Entanglement Entropy in Conformal Field Theory

In this chapter, we derive an expression for the entanglement entropy in $(1+1)$-dimensional conformal field theory. We will find a logarithmic relationship between the length of an interval and the maximum entropy it can harbor. This result will prove critical in Chapter 6 as we draw the connection between entanglement entropy in CFT_2 and geodesics in AdS_3.

4.1 Path Integral Representation

Let us adopt the abbreviation: $\phi_0(x) \equiv \phi(t=0,x)$. A wave functional Ψ has a path integral representation given by:

$$\Psi\left[\phi_0(x)\right] = \langle\phi_0(x)|\Psi\rangle = \int_{t=-\infty}^{t=0,\,\phi_0(x)} \left[\mathcal{D}\phi(t,x)\right] exp\left(\frac{i}{\hbar}S[\phi]\right) \tag{4.1}$$

Here, $S[\phi]$ denotes the *action* associated with the field and is not to be confused with entropy. Similarly, the conjugate wave functional is given by:

$$\Psi^*\left[\phi_0'(x)\right] = \langle\Psi|\phi_0'(x)\rangle = \int_{t=0,\phi_0'(x)}^{t=\infty} \left[\mathcal{D}\phi(t,x)\right] exp\left(\frac{i}{\hbar}S[\phi]\right) \tag{4.2}$$

The partition function $Z = \langle\Psi|\Psi\rangle$ is computed by performing a path integral over the entire space at a given moment in time ($t = 0$ in this case). This amounts to inserting the complete set of basis states $\int \mathcal{D}\left[\phi_0(x)\right] |\phi_0\rangle \langle\phi_0|$ as follows:

$$Z = \int \mathcal{D}\left[\phi_0(x)\right] \langle\Psi|\phi_0\rangle \langle\phi_0|\Psi\rangle = \int \mathcal{D}\left[\phi_0(x)\right] \Psi^*\left[\phi_0(x)\right]\ \Psi\left[\phi_0(x)\right] \tag{4.3}$$

Now suppose that we divide our spatial interval into two segments, A and B. The state $|\Psi\rangle$ of the entire system can be decomposed into the direct product $\left|\Psi^A\right\rangle\left|\Psi^B\right\rangle$[1], representing the portions of $|\Psi\rangle$ in regions A and B separately. The partial density matrix ρ_A is given by $\frac{1}{Z}\left|\Psi^A\right\rangle\left\langle\Psi^A\right|$. Inserting a complete set of basis states, we have [17]:

$$\begin{aligned}\rho_A &= \frac{1}{Z}\int \mathcal{D}\left[\phi_0^B(x\in B)\right]\left\langle\phi_0^B\middle|\Psi_B\right\rangle|\Psi_A\rangle\langle\Psi_A|\left\langle\Psi_B\middle|\phi_0^B\right\rangle \\ &= \frac{1}{Z}\int \mathcal{D}\left[\phi_0^B(x\in B)\right]\left\langle\phi_0^B\middle|\Psi\right\rangle\left\langle\Psi\middle|\phi_0^B\right\rangle \quad (4.4)\end{aligned}$$

This matrix is a path integral that traces over the states of ϕ_0 in region B but not region A. Recall that ϕ_0 denotes ϕ at $t=0$ and that $\langle\phi_0|\Psi\rangle$ and $\langle\Psi|\phi_0\rangle$ denote the path integrals of ϕ from $t=-\infty$ to $t=0^-$ and from $t=0^+$ to $t=\infty$, respectively. We now consider a matrix element of ρ_A, denoted:

$$[\rho_A]_{ab} \equiv \left\langle\phi_a^A\middle|\,\rho_A\,\middle|\phi_b^A\right\rangle = \frac{1}{Z}\int \mathcal{D}\left[\phi_0^B(x\in B)\right]\left\langle\phi_a^A\middle|\left\langle\phi_0^B\middle|\Psi\right\rangle\left\langle\Psi\middle|\phi_0^B\right\rangle\middle|\phi_b^A\right\rangle \quad (4.5)$$

where $\phi_A^a \equiv \phi_A^a(t=0^-, x\in A)$ and $\phi_A^b \equiv \phi_A^b(t=0^+, x\in A)$ are boundary conditions of the field ϕ in region A at $t=0^-$ and $t=0^+$, respectively.[2] So with this integral representation, we can clearly see how the states in the entire spacetime picture have been traced over, with the exception of the states in the region A at the exact moment $t=0$. We now execute the integral over the entire $(1+1)$-dimensional spacetime:

$$[\rho_A]_{ab} = \frac{1}{Z}\int \mathcal{D}\left[\phi(t,x)\right] e^{iS[\phi]/\hbar}\cdot\prod_{x\in A}\delta\left[\phi(0^+,x)-\phi_b^A(x)\right]\,\delta\left[\phi(0^-,x)-\phi_a^A(x)\right] \quad (4.6)$$

We now consider an expanded spacetime $\mathcal{M}_n$ created by "sewing" together n copies of this spacetime model, each with a discontinuous "cut" along A. The density matrix for this state is ρ_A^n. Let $Z_n \equiv Z[\mathcal{M}_n]$ denote the partition function for this expanded spacetime. Then we have:

$$tr_A(\rho_A^n) = \frac{Z_n}{(Z)^n} \quad (4.7)$$

Now, with the value for S_A derived in equation (3.16) as part of theorem 3.1.2, we have [17]:

$$S_A = -\lim_{n\to 1}\frac{\partial}{\partial n}\left(ln\left[\frac{Z_n}{(Z)^n}\right]\right) = -\lim_{n\to 1}\frac{\partial}{\partial n}(ln\, Z_n - n\, ln\, Z) \quad (4.8)$$

[1]The A and B here are indices denoting the portions of the field ϕ located in regions A and B. They are not exponents. All superscripts labelled A, B, a, or b in this chapter are indices, not exponents, unless otherwise stated.

[2]The analogous "boundary conditions" for region B would simply be $\phi_0^B(x)$, which is precisely the quantity over which we are integrating.

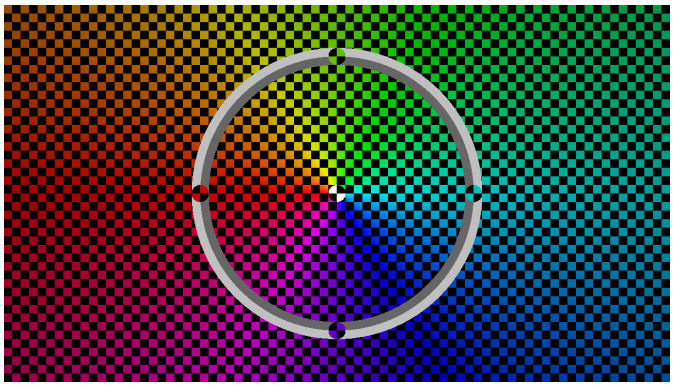

Figure 4.1: The complex plane. Note the black & white symbol at the origin and the gray unit circle. The light blue, green, red, and purple symbols which lie on the unit circle represent the numbers $l/2$, $il/2$, $-l/2$, and $-il/2$, respectively.

4.2 An Interval in the Complex Plane

We consider an interval of length l in the complex plane. This interval is situated along the real axis such that its endpoints lie at $z = -l/2$ and $l/2$.

4.2.1 Conformal Transformations

We now execute a pair of conformal transformations to make the analysis simpler. First, define a transformation $\zeta : \mathbb{C}\backslash\{z = l/2\} \to \mathbb{C}$ as follows [17]:

$$\zeta \equiv \frac{z + l/2}{z - l/2} \tag{4.9}$$

This maps the semi-open interval $[-l/2, l/2)$ to the semi-infinite line $(-\infty, 0]$ and "flips" the interval in the process. See figures 4.1 and 4.2.

We now execute another conformal transformation from the complex plane to the cylinder $w : \mathbb{C} \to \mathbb{R} \times S^1$ given by [17]:

$$w \equiv \tau + i\varphi = \frac{L}{2\pi} ln\, \zeta \tag{4.10}$$

where $\tau, \varphi \in \mathbb{R}$ and L is the circumference of the cylinder. To take the logarithm of a complex number ζ, refer to (B.5). Notice that the behavior of φ is periodic, so we can restrict its domain to $0 \leq \varphi < L$ and impose periodic boundary conditions: $\varphi \sim \varphi + kl$ for $k \in \mathbb{Z}$. Also note how the interval in which we are interested behaves under this transformation: it maps from the semi-infinite line

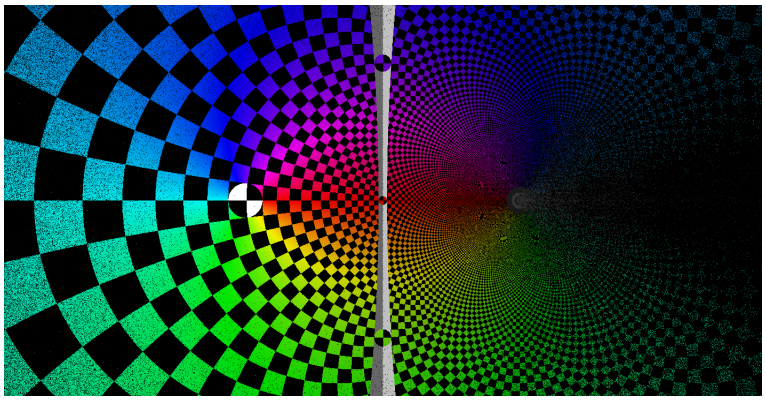

Figure 4.2: A graph depicting the complex mapping $\zeta(z) = \frac{z+l/2}{z-l/2}$. The origin gets mapped to -1, and the interval $(-l/2+a,\, l/2-a)$ gets flipped and mapped to $(-l/a,\, -a/l)$.

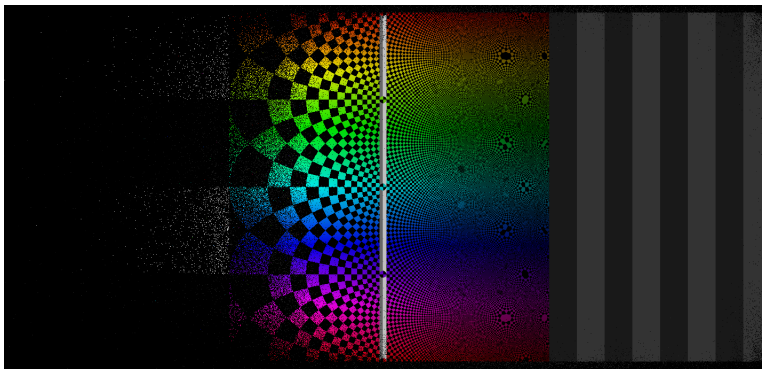

Figure 4.3: A graph depicting the complex mapping $w(z) = L/(2\pi)\, ln\, z$.

$(-\infty, 0]$ to another copy of the semi-infinite line $(-\infty, 0]$, scaling logarithmically in the process. See figures 4.3 and 4.4.

4.2.2 Imposing a Cutoff

It is well-known in quantum field theory that, if excitations are allowed to be unrestricted, then arbitrarily small-scale behavior generally yields a divergent contribution to physical quantities such as entropy. As this cannot physically be the case, we are compelled to introduce a manual cutoff to the ends of our interval, much in the same spirit that Max Planck introduced his famous constant to explain blackbody radiation. We ignore all points within a small radius a of either endpoint $z = \pm l/2$. Physically, a corresponds to the smallest

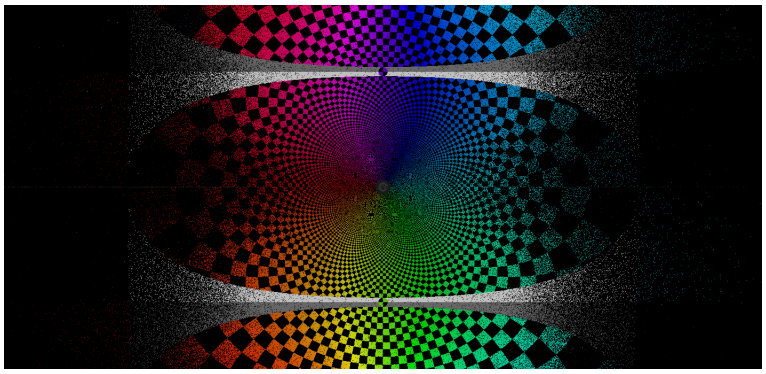

Figure 4.4: A graph depicting the combined complex mapping $w(\zeta(z)) = L/(2\pi)\ ln\left(\frac{z+l/2}{z-l/2}\right)$. This stretches out the original interval along the real (horizontal) axis, with enpoints at $\tau = \pm\frac{L}{2\pi}\ ln\left(\frac{l}{a}\right)$. It is on this transformed space which we perform the key entropy calculations.

scale at which an observer is capable of distinguishing a subsystem A within the interval with the remainder of the interval B, which should be thought of as the surroundings of A. At extremely small scales, the classical intuitive notions of space are no longer applicable anyway, so it is completely reasonable that a should have some finite value.

With this cutoff introduced, our interval now ranges from $-l/2+a$ to $l/2-a$. Under transformations (4.9) and (4.10), our interval gets mapped to the line at $\varphi = 0$ (the real number line), with endpoints at $\tau = \pm\frac{L}{2\pi}\ ln\left(\frac{l}{a}\right)$. Let us denote:

$$\beta \equiv \frac{L}{\pi}\ ln\left(\frac{l}{a}\right) \tag{4.11}$$

The choice to call this parameter β is deliberate, as it is meant to suggest the Boltzmann factor:

$$\beta = \frac{1}{k_B T} \tag{4.12}$$

for a classical thermodynamic system with temperature T and Boltzmann constant k_B, and:

$$\beta = \frac{it}{\hbar} \tag{4.13}$$

for a quantum system. Then the partition function of our cylinder is given by [17]:

$$Z[\mathcal{M}_n] = \langle 0|\, e^{-\beta H}\, |0\rangle \tag{4.14}$$

for ground state $|0\rangle$. The Hamiltonian H of this system, stated here without rigorous proof, is given by [17]:

$$H = \frac{2\pi}{nL}\left(\mathcal{L}_0 + \bar{\mathcal{L}}_0 - \frac{c}{12}\right) \tag{4.15}$$

where $\mathcal{L}_0$ and $\bar{\mathcal{L}}_0$ are creation and annihilation operators[3] that will cancel out when applied in formula (4.14) and c is the central charge of this particular conformal field theory. We now solve for the logarithm of the partition function by plugging in our results from formulas (4.11) and (4.15):

$$ln\, Z[\mathcal{M}_n] = -\langle 0|\, \beta H\, |0\rangle = \langle 0| \left[\frac{c}{6n} ln\left(\frac{l}{a}\right)\right] |0\rangle = \frac{c}{6n} ln\left(\frac{l}{a}\right) \tag{4.16}$$

We now plug this value into equation (4.8) to solve for the entanglement entropy [17]:[4]

$$\begin{aligned} S_A &= -\lim_{n\to 1} \frac{\partial}{\partial n}\left[\frac{c}{6n} ln\left(\frac{l}{a}\right) - n\cdot\frac{c}{6} ln\left(\frac{l}{a}\right)\right] = -\frac{c}{6}\, ln\left(\frac{l}{a}\right) \lim_{n\to 1} \frac{\partial}{\partial n}\left[\frac{1}{n} - n\right] \\ &= -\frac{c}{6}\, ln\left(\frac{l}{a}\right) \lim_{n\to 1}\left[\frac{-1}{n^2} - 1\right] = -\frac{c}{6}\, ln\left(\frac{l}{a}\right)[-1-1] = \frac{c}{3}\, ln\left(\frac{l}{a}\right) \end{aligned} \tag{4.17}$$

This is a very important result. It agrees with the holographic principle, which states that in a one-dimensional space, the maximum entropy capacity should vary logarithmically with the length of an interval. This result will also be of great importance in the following chapter as we look to draw the relationship between AdS and CFT.

4.3 Entropy Results

Our final result for the entanglement entropy of subsystem A is [3]:

$$S_A = \frac{c}{3}\, ln\left[\frac{L}{\pi a}\, sin\left(\frac{\pi l}{L}\right)\right] \tag{4.18}$$

where l is the length of subsystem A, L is the length of the total system $A \cup B$, a is the ultraviolet cutoff (lattice spacing), and c is the central charge of the CFT. In the case where subsystem A is very small relative to the entire system (that is, $l \ll L$), then we can apply the small angle approximation (B.23b) formula, obtaining:

$$S_A = \frac{c}{3}\, ln\left[\frac{l}{a}\right] \tag{4.19}$$

These results will be of vital importance in the next chapter where we will draw the relationship between AdS and CFT.

[3]These creation and annihilation operators are not to be confused with the Lagrangian density $\mathcal{L}$.

[4]Note that $Z[\mathcal{M}_n]$ is simply Z in the special case where $n = 1$.

Chapter 5: Srednicki's Theorem & Algorithm

Before turning to the Ryu-Takayanagi Proposal on the AdS/CFT correspondence, we devote a chapter to the contributions of Mark Srednicki to the subject of entanglement entropy and holography. We flesh out a theorem which suggests that the entanglement entropies of complementary subsystems are identical, leading us to the conclusion that entropy cannot be an extensive quantity. That is, $S \sim R^2$ for a three-dimensional region. We then describe a matrix algorithm for estimating the entanglement entropy of a quantum field by treating it as a series of quantum harmonic oscillators. The author has built Srednicki's matrix algorithm using MATLAB to show that, for a one-dimensional region in CFT, the entanglement entropy of a quantum field appears to vary logarithmically with the length of the interval, for sufficiently small intervals.

5.1 Srednicki's Holography Theorem

Let us consider a system consisting of a free, massless, scalar quantum field residing in a large spherical region (in ordinary 3-dimensional Euclidean space) of radius R_{out}. Let subsystem A consist of the field as restricted to a concentric sphere of radius R_{in} ($R_{in} < R_{out}$). Naturally, the surroundings B to this subsystem consist of the field at radii larger than R_{in} but less than R_{out}.[1] If we like, we can make R_{out} extremely large – the surroundings B can effectively represent the entire model universe except for the region inside the inner sphere. We seek to calculate the respective von Neumann entropies S_{in} and S_{out} of the inner and outer regions. Of critical importance here is the relationship between the entropy S and the radius R of a region of interest. We will find that $S \sim R^2$ in three-dimensional space. In general, $S \sim R^{d-1}$ in d-dimensional space, with

[1]The reader should note that, in this chapter, quantities associated with subsystem A will often be designated with the subscript *in* to emphasize that subsystem A resides inside the inner sphere; similarly, quantities associated with the surroundings B will often often be designated with the subscript *out*; generic quantities will not be denoted with subscripts.

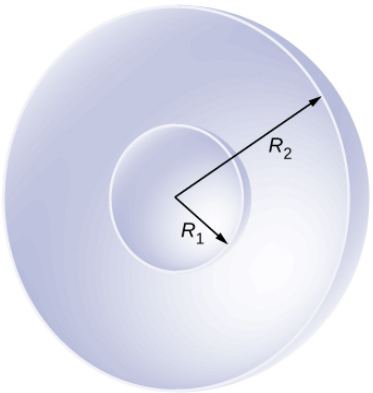

Figure 5.1: The small sphere divides the overall region inside the large sphere into inner and outer components. We seek to calculate the maximum entropy in each region.

$S \sim log\, R$ in the special case when $d = 1$.[2] See figure 5.1.

Theorem 5.1.1. $S_{in} = S_{out}$

Proof. Let $|0\rangle$ denote the non-degenerate normalized ground state of the entire system and let $\rho = |0\rangle\langle 0|$ be the ground state density operator. Equations (3.11a) and (3.11b) now read:

$$\begin{cases} S_{in} = -tr_{in}(\rho_{in}\, log\, \rho_{in}) \quad \rho_{in} = tr_{out}\left(|0\rangle\langle 0|\right) & \text{(5.1a)} \\ S_{out} = -tr_{out}(\rho_{out}\, log\, \rho_{out}) \quad \rho_{out} = tr_{in}\left(|0\rangle\langle 0|\right) & \text{(5.1b)} \end{cases}$$

It is important to note that $|0\rangle$ itself consists of a direct product of ket vectors $|0\rangle_{in}|0\rangle_{out}$, representing the substates of A and B, respectively. We can expand this as: $|0\rangle = \sum_{i,a} \psi_{ia}|i\rangle_{in}|a\rangle_{out}$, for some tensor-like quantity ψ_{ia}. The corresponding bra vector is given by: $\sum_{j,b}\langle j|_{in}\langle b|_{out}\psi^*_{bj}$.[3,4] In this notation, we have:

[2] It should be noted, however, that these results are most applicable when $R_{in} \ll R_{out}$, as this allows us to treat R_{out} as effectively infinite.

[3] The choice of index names is not important, so long as indices that vary independently of one another have distinct names.

[4] A coefficient paired with a bra vector corresponds to a transposed matrix.

$$\rho_{in} = tr_{out}\left(\sum_{i,a}\sum_{j,b}\psi_{ia}\left|i\right\rangle_{in}\left|a\right\rangle_{out}\left\langle j\right|_{in}\left\langle b\right|_{out}\psi^*_{bj}\right)$$
$$= \sum_{i,j,a}\psi_{ia}\left|i\right\rangle_{in}\left\langle j\right|_{in}\left|a\right\rangle_{out}\left\langle a\right|_{out}\psi^*_{aj} = \sum_{i,j,a}\psi_{ia}\left|i\right\rangle_{in}\left\langle j\right|_{in}\psi^*_{aj} \quad (5.2)$$

$$\rho_{out} = tr_{in}\left(\sum_{i,a}\sum_{j,b}\psi_{ia}\left|i\right\rangle_{in}\left|a\right\rangle_{out}\left\langle j\right|_{in}\left\langle b\right|_{out}\psi^*_{bj}\right)$$
$$= \sum_{i,a,b}\psi_{ai}\left|i\right\rangle_{in}\left\langle i\right|_{in}\left|a\right\rangle_{out}\left\langle b\right|_{out}\psi^*_{ib} = \sum_{i,a,b}\psi_{ai}\left|a\right\rangle_{out}\left\langle b\right|_{out}\psi^*_{ib} \quad (5.3)$$

where our notation is such that the order of the indices is always such that the second index of the first matrix element matches the first index of the second matrix element so that the matrix multiplication comes out cleanly. This requires us to switch the order of the indices of ψ and ψ^* in calculating ρ_{out}, meaning the matrices are being transposed. The result is:

$$(\rho_{in})_{ij} = \left(\psi\psi^\dagger\right)_{ij} \text{ or } \rho_{in} = \psi\psi^\dagger \quad (5.4a)$$
$$(\rho_{out})_{ab} = \left(\psi^T\psi^*\right)_{ab} \text{ or } \rho_{out} = \psi^T\psi^* \quad (5.4b)$$

Consider the matrix $\rho^*_{out} = (\psi^T\psi^*)^* = \psi^\dagger\psi$. By (C.9) and (C.10), the matrices ρ_{in}, ρ_{out}, and ρ^*_{out} must all be Hermitian and hence their eigenvalues must all be real numbers. By (C.11), ρ^*_{out} must have the same set of eigenvalues as ρ_{out} and hence the same trace. Applying all these properties along with (C.3) together, we conclude:

$$tr(\rho_{in}) = tr(\psi\psi^\dagger) = tr(\psi^\dagger\psi) = tr(\rho^*_{out}) = tr(\rho_{out}) \quad (5.5)$$

This implies that ρ_{in} and ρ_{out} have the same eigenvalues (if one of them has a higher rank, then its remaining eigenvalues will be extra zeros). From (3.4), we have:

$$S_{in} = -tr_{in}(\rho_{in}\,log\,\rho_{in}) = -\sum_x \lambda^{in}_x\,log\,\lambda^{in}_x \quad (5.6)$$

$$S_{out} = -tr_{out}(\rho_{out}\,log\,\rho_{out}) = -\sum_x \lambda^{out}_x\,log\,\lambda^{out}_x \quad (5.7)$$

But since ρ_{in} and ρ_{out} have the same non-zero eigenvalues, we can conclude:

$$S_{in} = S_{out} \quad (5.8)$$

■

This is an astonishing result. It says that the total entropy outside of subsystem A is equivalent to the entropy inside of subsystem A, regardless of the size of A. A could have an extremely small volume and yet this conclusion would still hold. How can this be? Though this is not yet a definitive proof of the intimate relationship between entropy and area, this result is quite suggestive. The *two-dimensional boundary* between subsystem A and surroundings B is the only common element of the two regions, regardless of their relative sizes. This suggests that the area of this *boundary* is proportional to the entropy of each region.

5.2 The Srednicki Entropy Algorithm

5.2.1 Two Coupled Harmonic Oscillators

Consider a system of two coupled one-dimensional harmonic oscillators, which we will informally refer to as 1 and 2. Each oscillator is tethered to a fixed wall by a spring of stiffness k_0, and the oscillators are tethered to one another by a spring with stiffness k_1. For simplicity, let us set the mass of all oscillators equal to 1. Let x_1 denote the displacement from equilibrium for oscillator 1 and x_2 denote the displacement from equilibrium for oscillator 2. Even without knowledge of quantum mechanics, one can write down the Hamiltonian of this system:

$$H = \frac{1}{2}\left[p_1^2 + p_2^2 + k_0(x_1^2 + x_2^2) + k_1(x_1 - x_2)^2\right] \tag{5.9}$$

Before solving the Schrödinger equation, let us first define a few auxiliary quantities:

$$x_\pm \equiv \frac{x_1 \pm x_2}{\sqrt{2}}, \quad \omega_+ \equiv \sqrt{k_0}, \quad \omega_- \equiv \sqrt{k_0 + 2k_1} \tag{5.10}$$

We now solve the Schrödinger equation and find the ground state wave function:

$$\psi_0(x_1, x_2) = \left(\frac{\omega_+\omega_-}{\pi^2}\right)^{1/4} \exp\left(-\frac{\omega_+ x_+^2 + \omega_- x_-^2}{2}\right) \tag{5.11}$$

Since this wave function is completely real, its complex conjugate is an identical copy, which we will denote using primed versions of our position variables: $\psi_0^*(x_1', x_2')$. We now treat oscillator 1 as "inside" and oscillator 2 as "outside." The ground state density matrix ρ is of course formed by taking the trace over the entire system, but ρ_{out} is formed by taking the partial trace over the "inside" component of the system:

$$\rho_{out}(x_2, x_2') = \int_{-\infty}^{\infty} \psi_0(x_1, x_2)\psi_0^*(x_1, x_2')dx_1 \tag{5.12}$$

Before carrying out this integral, let us define a few more auxiliary quantities:

$$\beta \equiv \frac{(\omega_+ - \omega_-)^2}{4(\omega_+ + \omega_-)}, \quad \gamma \equiv \frac{2\omega_+\omega_-}{\omega_+ - \omega_-} + \beta \tag{5.13}$$

Carrying out this integral on the wave function given by (5.11), the result is:

$$\rho_{out}(x_2, x_2') = \sqrt{\frac{\gamma - \beta}{\pi}} \exp\left(-\gamma \frac{x_2^2 + {x_2'}^2}{2} + \beta x_2 x_2'\right) \tag{5.14}$$

Let λ_n and $f_n(x)$ respectively denote the eigenvalues and eigenfunctions of the operator ρ_{out}. That is:

$$\int_{-\infty}^{\infty} \rho_{out}(x, x') f_n(x') dx' = \lambda_n f_n(x) \tag{5.15}$$

Once again, let us define a few more auxiliary quantities before solving for λ_n and $f_n(x)$:

$$\alpha \equiv \sqrt{\gamma^2 - \beta^2} = \sqrt{\omega_+\omega_-}, \quad \xi = \frac{\beta}{\gamma + \alpha} \tag{5.16}$$

The solution to (5.15) is given by:

$$\lambda_n = (1 - \xi)\xi^n, \quad H_n(x\sqrt{\alpha})e^{-\alpha x^2/2} \tag{5.17}$$

where H_n is the n^{th} degree Hermite polynomial, with argument $x\sqrt{\alpha}$. Of course, the eigenvalues are of great interest because they enter directly into the formula for the entanglement entropy. Plugging in the appropriate values for λ_n into formula (3.4), we have:

$$S = -\sum_{n=0}^{\infty} (1 - \xi)\xi^n ln\left[(1 - \xi)\xi^n\right] \tag{5.18}$$

We can simplify this expression using formulas (B.3), (B.4), (D.3), and (D.5) to obtain the following key result:

$$S = -ln(1 - \xi) - \frac{\xi}{1 - \xi} ln(\xi) \tag{5.19}$$

This result is extremely useful. Note that ξ is defined purely in terms of the spring constants k_0 and k_1.

5.2.2 N Coupled Harmonic Oscillators

We now seek to generalize the result obtained in the previous section to a more complex system of N coupled harmonic oscillators. Such a system is described with a Hamiltonian:

$$H = \frac{1}{2}\sum_{i=1}^{N} {p_i}^2 + \frac{1}{2}\sum_{i,j=1}^{N} x_i K_{ij} x_j \tag{5.20}$$

for a real symmetric matrix K with positive eigenvalues. Since K is real and symmetric, we can express it as $K = U^T K_D U$ for diagonal matrix K_D and orthogonal matrix U. Let us now define:

$$\Omega \equiv U^T K_D^{1/2} U \tag{5.21}$$

As one can easily verify, $\Omega^2 = K$. Let us adopt the abbreviation: $x \equiv (x_1, \ldots, x_N)$. With these definitions, we can write the ground state wave function of the Hamiltonian as [22]:

$$\psi_0(x) = \pi^{-N/4} \left(\det \Omega\right)^{1/4} \exp\left[-\frac{x \cdot \Omega \cdot x}{2}\right] \tag{5.22}$$

where the dot represents matrix-vector multiplication.[5]

Let us now divide our full set of oscillators into two disjoint subsets: denote oscillators 1 to n the "inside" oscillators and oscillators $(n+1)$ to N the "outside" oscillators. Let us also adopt the abbrevations: $x_{in} \equiv (x_1, \ldots, x_n)$ and $x_{out} \equiv (x_{n+1}, \ldots, x_N)$. Of course, $x = x_{in} \otimes x_{out}$. Subsequent vector definitions will have related definitions. We decompose the matrix Ω into submatrices as follows:

$$\Omega = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \tag{5.23}$$

where A is $n \times n$, B is $n \times (N-n)$, and C is $(N-n) \times (N-n)$. We wish to compute ρ_{out} by tracing over the first n oscillators:

$$\rho_{out}(x_{out}; x'_{out}) = \int \prod_{i=1}^{n} dx_i \psi_0(x_{in} \otimes x_{out}) \times \psi_0^*(x_{in} \otimes x'_{out}) \tag{5.24}$$

Before computing ρ_{out}, let us define the following auxiliary quantities $\beta \equiv \frac{1}{2} B^T A^{-1} B, \quad \gamma \equiv C - \beta$. With these definitions, we have [22]:

$$\rho_{out}(x_{out}; x'_{out}) \sim \exp\left[-\frac{(x_{out} \cdot \gamma \cdot x_{out} + x'_{out} \cdot \gamma \cdot x'_{out})}{2} + x_{out} \cdot \beta \cdot x'_{out}\right] \tag{5.25}$$

We now diagonalize γ and using the appropriate change of basis matrix V: $\gamma = V^T \gamma_D V$.[6] We now define a new vector y_{out} as: $y_{out} \equiv x_{out} V^T \gamma_D^{1/2}$. We define

[5] There is a unique way here in which one can arrange the rows and columns of these vectors and matrices such that their multiplication is compatible: the first x must be a $1 \times n$ row vector, Ω is an $n \times n$ matrix, and the second x must be an $n \times 1$ column vector.

[6] γ_D is diagonal and the change of basis matrix V is orthogonal.

a new matrix β' as: $\beta' \equiv \gamma_D^{-1/2} V \beta V^T \gamma_D^{-1/2}$. We diagonalize β' as follows: $\beta' = W \beta'_D W^T$ for orthogonal matrix W. We now define $z_{out} \equiv W^T y_{out}$. Finally, let β'_i denote an eigenvalue of β'. With all these definitions, we can now express ρ_{out} in terms of z_{out}:

$$\rho_{out}(z_{out}, z'_{out}) \sim \prod_{i=1}^{N-n} \exp\left[-\frac{(z_{out})_i^2 + (z'_{out})_i^2}{2} + \beta'_i (z_{out})_i (z'_{out})_i\right] \tag{5.26}$$

Notice the resemblance to equation (5.14), the outside density matrix for the $N = 2$ case. This resemblance means that the von Neumann entropy for arbitrary N is an expansion upon the entropy for the $N = 2$ case. Namely [22]:

$$S = \sum_i S(\xi_i), \text{with } S(\xi) \text{ given by (5.19), and } \quad \xi = \frac{\beta'_i}{1 + \sqrt{1 - (\beta'_i)^2}} \tag{5.27}$$

This is a very general result and is the foundation for the major original contribution of this paper, which will be detailed in the following section.

5.3 Entropy Computations from the Srednicki Algorithm

With the algorithm detailed in the previous section, Srednicki has laid the groundwork for entropy calculations of more general systems. Note that the entropy is a quantity that is ultimately calculable from just two quantities: K, the real symmetric matrix which comprehensively describes the interactions amongst the N harmonic oscillators (K is $N \times N$-dimensional), and n, the number of oscillators which are considered "inside" and are hence traced over.

5.3.1 One-Dimensional Interval

In this paper, we are chiefly concerned with the entanglement entropy harbored by a one-dimensional spatial interval of length L due to a massless scalar quantum field ϕ in its ground state. For such a system, the Hamiltonian H and Hamiltonian density $\mathcal{H}$ are given by:

$$H = \int \mathcal{H}\, dx = \frac{1}{2} \int \left(\pi^2(x) + |\boldsymbol{\nabla}\phi(x)|^2\right) dx \tag{5.28}$$

where $\pi(x)$ is the conjugate momentum field of $\phi(x)$, with canonical commutation relation:

$$[\phi(x), \pi(x')] = i\delta(x - x') \tag{5.29}$$

Of course, in one dimension, the gradient simply reduces to an ordinary derivative in x: $\boldsymbol{\nabla}\phi(x) = \frac{\partial}{\partial x}\phi(x)$. To discretize the field, we introduce a lattice spacing a such that $(N+1)\,a = L$. We then denote the field value at each point k by ϕ_k, for $k = 0, 1, 2, \ldots, N, (N+1)$; $k = 0$ denotes the inside extreme end and $k = N+1$ denotes the outside extreme end. We impose boundary conditions, demanding that the field vanish at either extreme end: $\phi_0 = \phi_(N+1)$. The discrete version of the derivative is:

$$\frac{\partial}{\partial x}\phi(x_k) \;\longrightarrow\; \frac{\phi(x_k) - \phi(x_{k-1})}{a} \tag{5.30}$$

Following the Hamiltonian model of Srednicki (5.20), we see that $\pi(x_k)$ here plays the role of $_i$ and $\phi(x_k)$ here plays the role of x_i. We must now determine the elements of the matrix K:

Theorem 5.3.1. *For a system of periodic boundary conditions with N lattice points and lattice spacing a, K is (up to a numerical factor of a) given by an N-dimensional matrix such that all entries on the principal diagonal are 2, all entries on either of the neighboring diagonals are -1, and all other elements are 0. That is:*

$$K = \begin{bmatrix} 2 & -1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2 \end{bmatrix} \tag{5.31}$$

Proof. Let us adopt the shorthand notation $\phi_k \equiv \phi(x_k)$. We want to draw a connection between the summation Hamiltonian (5.20) and the integral Hamiltonian (5.28). Just focusing on the part that involves matrix K, we have a correspondence:

$$\frac{1}{2}\sum_{i,j=1}^{N} x_i K_{ij} x_j \longleftrightarrow \frac{1}{2}\int |\boldsymbol{\nabla}\phi(x)|^2 dx \tag{5.32}$$

Now, we discretize the integral by replacing it with a sum and using our discretized derivative formula (5.30):[7]

$$\frac{1}{2}\sum_{i,j=1}^{N} x_i K_{ij} x_j \;\longleftrightarrow\; \frac{1}{2}\sum_{k=1}^{N+1}\left(\frac{\phi_k - \phi_{k-1}}{a}\right)^2 a \tag{5.33}$$

[7] We increase the summation to include the $(N+1)^{th}$ lattice point so as to be symmetric on both ends of the interval. Recall that $\phi_{N+1} = 0$ anyway.

where a, the lattice spacing between points, has now taken the role of the dx from the integral. Expanding out the right-hand side, we have:[8]

$$\begin{aligned}
&\frac{a}{a^2}\left[(\phi_1-\phi_0)^2+(\phi_2-\phi_1)^2+\cdots+(\phi_N-\phi_{N-1})^2+(\phi_{N+1}-\phi_N)^2\right] \\
&\quad=\frac{1}{a}\big[(\phi_0^2-\phi_0\phi_1-\phi_1\phi_0+\phi_1^2)+(\phi_1^2-\phi_1\phi_2-\phi_2\phi_1+\phi_2^2)+\ldots \\
&+(\phi_{N-1}^2-\phi_N\phi_{N-1}-\phi_{N-1}\phi_N+\phi_N^2)+(\phi_N^2-\phi_N\phi_{N+1}-\phi_{N+1}\phi_N+\phi_{N+1}^2)\big] \\
&\quad=\frac{1}{a}\big[\phi_0^2-\phi_0\phi_1-\phi_1\phi_0+2\phi_1^2-\phi_1\phi_2-\phi_2\phi_1+2\phi_2^2+\ldots \\
&+2\phi_{N-1}^2-\phi_{N-1}\phi_N-\phi_N\phi_{N-1}+2\phi_N^2-\phi_N\phi_{N+1}-\phi_{N+1}\phi_N+\phi_{N+1}^2\big]
\end{aligned} \tag{5.34}$$

Now, with periodic boundary conditions, we can set $\phi_0=\phi_{N+1}=0$. The result is:

$$\begin{aligned}
&\frac{1}{a}\big[2(\phi_1^2+\phi_2^2\cdots+\phi_N^2)-\phi_1\phi_2-\phi_2\phi_1-\phi_2\phi_3-\phi_3\phi_2-\ldots \\
&\qquad\qquad -\phi_{N-1}\phi_N-\phi_N\phi_{N-1}\big]
\end{aligned} \tag{5.35}$$

Note the pattern is exactly such that the this quantity (up to a factor of a) could be reproduced by executing the matrix multiplication: $\phi K\phi^T$, where K is as given above and ϕ represents the row vector: $(\phi_1,\ldots,\phi_N)$.

■

5.3.2 Original Computations of Entanglement Entropy Using MATLAB

Using the algorithm described by Srednicki, one can build computer code to perform the matrix calculations, which become increasingly rigorous for greater values of N. The main body of the code can be applied to any system, as long as the appropriate K matrix is selected. In this case, we adopt K as described in the previous section. For an arbitrary system, one must derive the K matrix from the Hamiltonian H. See figure 5.2.

The algorithm was run for $N=400$ and the results are plotted in figure 5.3. The data generated from this algorithm closely follow a logarithmic curve, particularly when n is less than about 200, or half N. The deviation from logarithmic behavior at high values for n is not worrisome because, as we have argued, the holographic principle is most applicable when the "inside" subsystem (in this case the n lattice points traced over) is considerably smaller than the "outside" subsystem (the remainder of the points), such that the "outside"

[8] Since all fields ϕ are not ordinary numbers but rather *operators*, we do not make the potentially naive assumption that field values at different points in space commute.

```
function [S] = Entropy(N,n)
% Returns the entropy of a 1D system tracing over the first n sites with N
% nontrivial lattice points

% Step 1: Build K
K = zeros(N);
K(1,1) = 2;
for i = 2:N
    K(i,i-1) = -1;
    K(i-1,i) = -1;
    K(i,i) = 2;
end

% Step 2: Diagonalize K
[U, K_D] = eig(K);
U_T = U';

% Step 3: Find Omega
Omega = U_T*sqrtm(K_D)*U;

% Step 4: Decompose Omega
A = Omega(1:n,1:n);
B = Omega(1:n,(n+1):N);
B_T = Omega((n+1):N,1:n);
C = Omega((n+1):N,(n+1):N);

% Step 5: Build auxiliary quantities
beta = 1/2*B_T*inv(A)*B;
gamma = C - beta;

% Step 6: Diagonalize gamma
[V, gamma_D] = eig(gamma);
V_T = V';

% Step 7: Build more auxiliary quantities
beta_prime = inv(sqrtm(gamma_D))*V*beta*V_T*inv(sqrtm(gamma_D));

% Step 8: Find the eigenvalues of beta_prime
beta_prime_i = eig(beta_prime);

% Step 9: Compute xi_i
xi_i = beta_prime_i./[1+(1-beta_prime_i.^2).^(1/2)];

% Step 10: Compute S_i, a vector whose elements are S(xi_i)
S_i = -log(ones(N-n,1)-xi_i) - xi_i./(1-xi_i).*log(xi_i);

% Step 11: The total entropy is simply the sum of all S_i
S = sum(S_i);
end
```

Figure 5.2: Original MATLAB algorithm to compute the entanglement entropy of a free, massless, scalar quantum field in a one-dimensional interval, modeled as a series of coupled harmonic oscillators

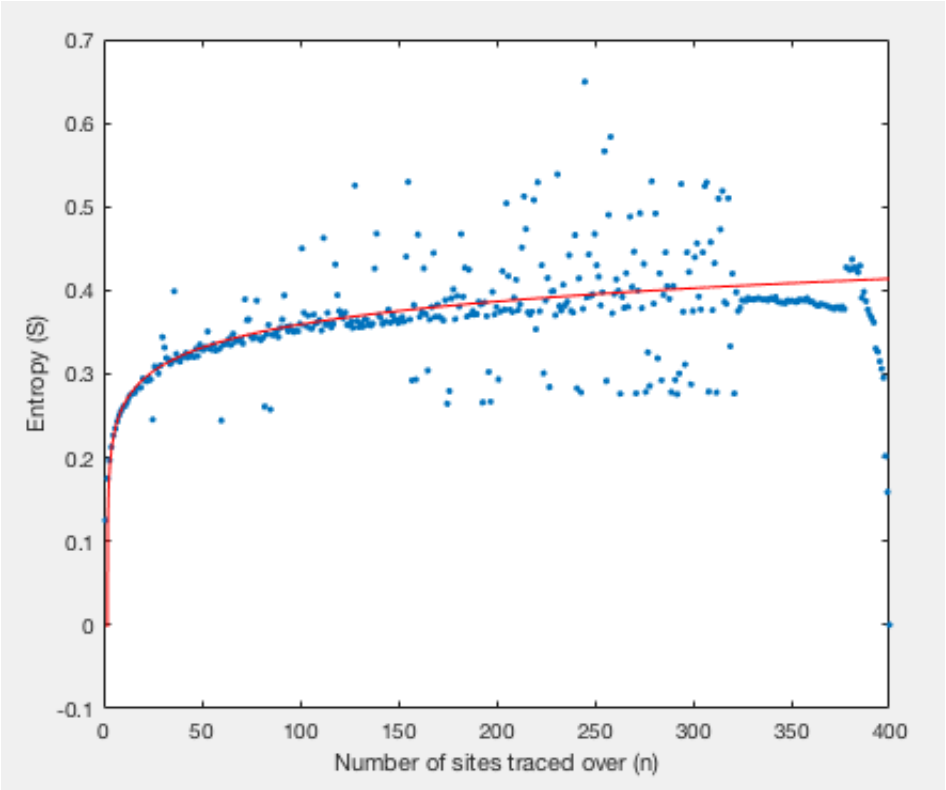

Figure 5.3: Computed values for entanglement entropy (S) as a function of number of sites traced over (n), for total number of nontrivial lattice sites $N = 400$. The data have been fit to a logarithmic model, represented by the solid red curve.

subsystem can be approximated as infinitely large. In fact, it is not at all surprising that the data points rapidly descend at very high values of n; we expect the entropy of the entire system to be 0 because the entire system is in a pure state.

Chapter 6: The Ryu-Takayanagi Proposal, the AdS_3/CFT_2 Correspondence, and Holography

We have now come to the point where we can tie together the material from previous chapters, establishing a connection between anti-de Sitter space and conformal field theory. In 1998, Juan Maldacena first proposed this AdS/CFT correspondence as an example of the holographic principle [14]. Research into this subject area has been ballooning ever since.

The argument given by Ryu and Takayanagi is valid in an arbitrary number of dimensions, so long as the number of spacetime dimensions of the AdS is one greater than the number of spacetime dimensions of the CFT. To be consistent with our previous notation, we will let D be the number of spacetime dimensions in the AdS, and $D-1$ be the number of spacetime dimensions in the CFT. (Both the AdS and the CFT have exactly one temporal dimension, so d is the number of spatial dimensions in the AdS and $(d-1)$ is the number of spatial dimensions in the CFT.) While this study is particularly interested in the lowest-dimensional case ($d = 2, D = 3$), we will try to keep the argument general whenever possible.

6.1 The AdS/CFT Model

Consider a CFT on $\mathbb{R} \times \mathrm{S}^{d-1}$ and suppose a subsystem A has a $(d-2)$-dimensional boundary $\partial A \, \epsilon \, S^{d-1}$. Now, let γ_A be the unique $(d-1)$-dimensional static minimal curve in AdS_{d+1} with boundary ∂A.[1] In the $d = 2$ case, this model can be well represented by an upright cylinder whose boundary represents

[1] If $d = 2$, γ_A is a minimal curve and ∂A is simply a pair of endpoints; if $d = 3$, γ_A is a minimal surface and ∂A is its one-dimensional boundary; if $d = 4$, γ_A is a minimal volume and ∂A is its two-dimensional boundary; and so on.

the CFT_2 space and whose interior or "bulk" represents AdS_3. The cylinder's height is the (proper) time dimension, with horizontal circular cross-sections representing Poincaré disks. From differential geometry, it is known that a minimal curve on a two-dimensional surface is in fact a geodesic curve.[2] As we will prove in theorem 6.2.1, the boundary of this cylinder is a (1+1)-dimensional flat spacetime to which we can apply the results of CFT from previous chapters. Let L denote the circumference of the cylinder, which is also the length of the finite spatial interval studied in the previous chapter. The two-dimensional boundary is divided into two regions A and B along surfaces of constant θ.[3] Let l denote the length of subsystem A ($l \leq L$). Then the angular breadth of A is $2\pi l/L$ while the angular breadth of B is $2\pi(1 - l/L)$. These quantities can easily be translated into the AdS language of Chapter 2: subsystem A is centered around $\theta = \theta_b$ and has an angular breadth of 2α. Therefore, $\alpha = \pi l/L$.

6.2 Entropy as the Length of a Geodesic

According to Ryu and Takayanagi's hypothesis (and expressed in natural units):[4]

$$S_A = \frac{\text{Length of } \gamma_A}{4\, G_N^{(d+1)}} \tag{6.1}$$

where $G_N^{(d+1)}$ is a $(d+1)$-dimensional version of Newton's gravitational constant.[5] This formula establishes a clear link between CFT, represented by the entanglement entropy on the left-hand side, and AdS, represented by the length of geodesic γ_A on the right-hand side.

6.2.1 Cutoffs

As we saw in theorem 2.2.1, the AdS metric produces divergent values when the radial coordinate is allowed to increase without bound. This would mean that the length of any geodesic is infinite; according to equation (6.1), this would imply that the entropy of the CFT interval is also infinite. To avoid this nonsensical conclusion, we manually impose a cutoff on ρ: $\rho \leq \rho_0$, where ρ_0 is a

[2] That is, curves of minimal length have the property that their tangent vectors remain tangent to the curve when parallel transported along the curve.

[3] Recall the global coordinates (τ, ρ, θ) and the static coordinates (τ, r, θ).

[4] Of course, in the general d-dimensional case, the *length* of γ_A in formula (6.1) should be replaced with the appropriate word describing the size of a spatial parcel of dimension $d-1$.

[5] The precise value for such a constant does not have an *a priori* value unless $d = 3$, in which case it is the familiar $6.67 \cdot 10^{-11} \frac{N \cdot m^2}{kg^2}$. In an arbitrary number of dimensions, the value $G_N^{(d+1)}$ would need to be determined experimentally, just as Cavendish once did for our universe in the late 18^{th} century.

large but finite constant. This approximates the boundary as a two-dimensional region of large but finite size. In the dual CFT, this cutoff corresponds to the ultraviolet cutoff associated with the lattice spacing a, with the relation between these two cutoffs given by [21]:

$$e^{\rho_0} \sim L/a \tag{6.2}$$

Before we proceed further, let us verify that the 2-dimensional boundary spacetime is indeed flat, as the holographic principle demands.

Theorem 6.2.1. *The 2-dimensional boundary spacetime is nearly flat, with perfect flatness in the limiting case where* $\rho_0 \to \infty$.

Proof. Working in global coordinates (2.5), the boundary space is obtained simply by setting the radial coordinate $\rho = \rho_0$, hence $d\rho = 0$. The space now only has two coordinates: (τ, θ). The metric reads:

$$ds^2 = -cosh^2\rho_0 d\tau^2 + sinh^2\rho_0 d\theta^2 \tag{6.3}$$

where θ is now playing the role of the spatial coordinate σ is out (1+1)-dimensional CFT space. As long as our cutoff value ρ_0 is large,[6] we can take advantage of the following limit:

$$\lim_{\rho_0 \to \infty} tanh(\rho_0) = 1 \;\Rightarrow\; sinh(\rho_0) \approx cosh(\rho_0) \equiv U \tag{6.4}$$

With this new constant U defined in this way, we can re-write the spacetime metric in this limit as:

$$ds^2 = U^2(-d\tau^2 + d\theta^2) \tag{6.5}$$

It is now a simple matter of rescaling and renaming the spacetime coordinates to transform this into the famous Minkowski metric, the hallmark of a flat spacetime: $ds^2 = -dt^2 + dx^2$. Hence it is clear that the boundary of our anti-de Sitter space is nearly flat and that, in the limiting case, it becomes perfectly flat. ■

In Chapter 2, we rigorously proved that the bulk AdS_3 space is in fact curved, and hence the holographic principle can be applied to this AdS/CFT pair of spaces.[7]

[6] Recall ρ has no finite upper bound.

[7] Recall the holographic principle demands that one space have one more spacetime dimension than the other and that the higher-dimensional space be curved and the lower-dimensional space be flat.

6.2.2 The Regulated Length of a Geodesic

Now that we have imposed a cutoff on our radial coordinate value, we can compute the regulated length s of a geodesic. It is easiest to compute the desired result in Poincaré coordinates. Recall from equations (2.19) that we can write the geodesic as: $(x, z) = l/2(cos\, g, sin\, g)$, where $g = 2\, tan^{-1}(e^s)$. $z = 0$ corresponds to the omega points, so we impose the cutoff on g: $\epsilon \leq g \leq \pi - \epsilon$, where ϵ is related to the CFT lattice spacing a by: $\epsilon = 2a/l$. Note that g is defined such that $ds = dg/sin\, g$. So the regulated length is then given by:

$$\text{Length}(\gamma_A) = r_{sk} \int ds = r_{sk} \int_{\epsilon}^{\pi-\epsilon} \frac{dg}{sin\, g} = ln \left[tan \left(\frac{g}{2} \right) \right]_{\epsilon}^{\pi-\epsilon} \tag{6.6}$$

Now, evaluating the integral and applying the trigonometric identity (B.21c) and the logarithm multiplication rule (B.3), we have:

$$\text{Length}(\gamma_A) = r_{sk}\, ln \left[\frac{sin(\pi - \epsilon)}{cos(\pi - \epsilon) + 1} \right] - ln \left[\frac{sin\, \epsilon}{cos\, \epsilon\, + 1} \right] = r_{sk}\, ln \left[\frac{\frac{sin(\pi-\epsilon)}{cos(\pi-\epsilon)+1}}{\frac{sin\, \epsilon}{cos\, \epsilon + 1}} \right] \tag{6.7}$$

We now apply the reflected angle formulas (B.22c) and (B.22d), the small angle approximation (B.23d), and the logarithm power rule (B.4):

$$\begin{aligned} \text{Length}(\gamma_A) &= r_{sk}\, ln \left[\frac{\frac{sin\, \epsilon}{-cos\, \epsilon + 1}}{\frac{sin\, \epsilon}{cos\, \epsilon + 1}} \right] = r_{sk}\, ln \left[\frac{\cos\, \epsilon + 1}{-\cos\, \epsilon + 1} \right] \\ &\approx r_{sk}\, ln \left[\frac{1 - \epsilon^2/2 + 1}{-(1 - \epsilon^2/2) + 1} \right] \approx r_{sk}\, ln \left[\frac{2}{\epsilon^2/2} \right] = r_{sk}\, ln \left[\left(\frac{2}{\epsilon} \right)^2 \right] = 2\, r_{sk}\, ln \left[\frac{2}{\epsilon} \right] \end{aligned} \tag{6.8}$$

Since $\epsilon = 2a/l$, $2/\epsilon = l/a$. So we have:

$$\text{Length}(\gamma_A) = 2\, r_{sk}\, ln \left(\frac{l}{a} \right) \tag{6.9}$$

Now, according to the Ryu-Takayangi formula (6.1), the entanglement entropy S_A of a region A whose endpoints match those of this geodesic γ_A is given by:

$$S_A = \frac{r_{sk}}{2\, G_N^{(3)}}\, ln \left(\frac{l}{a} \right) \tag{6.10}$$

Notice that this result matches the conformal field theory result (4.19) if we demand that:

$$c = \frac{3 r_{sk}}{2 G_N^{(3)}} \tag{6.11}$$

This equation establishes a link between c, the central charge of the conformal field theory, and r_{sk}, the skirt radius of anti-de Sitter space.

A more exact formula for the regulated length s of a geodesic is given in terms of the global coordinates:

$$cosh(s/r_{sk}) = \left[1 + 2sinh^2\,\rho_0\,sin^2(\pi l/L)\right] \tag{6.12}$$

Theorem 6.2.2. *The entanglement entropy of subsystem A is given by [20]:*

$$S_A \simeq \frac{r_{sk}}{4G_N^{(3)}}\,ln\left[e^{2\rho_0}sin^2\left(\frac{\pi l}{L}\right)\right] = \frac{c}{3}\,ln\left[\frac{L}{a}sin\left(\frac{\pi l}{L}\right)\right] \tag{6.13}$$

Proof. We apply the approximation from theorem B.3.1 to equation (6.12):

$$\frac{s}{r_{sk}} \approx ln\,\left[2\,cosh\left(\frac{s}{r_{sk}}\right)\right] = ln\left[2\,\left(1 + 2sinh^2\,\rho_0\,sin^2\left[\frac{\pi l}{L}\right]\right)\right] \tag{6.14}$$

Now, from theorem B.3.1, we know we can approximate $sinh(\rho_0)$ by $e^{\rho_0}/2$. So we have:

$$\frac{s}{r_{sk}} \approx ln\left[2 + e^{2\rho_0}\,sin^2\left(\frac{\pi l}{L}\right)\right] \;\Rightarrow\; s \approx\, r_{sk}\,ln\left[e^{2\rho_0}\,sin^2\left(\frac{\pi l}{L}\right)\right] \tag{6.15}$$

where we have dropped the addition of 2 in the logarithm's argument as it is very small in comparison to the $e^{2\rho_0}$ term. We now plug this approximate result into the Ryu-Takayanagi formula (6.1) to obtain the entropy:

$$S_A \simeq \frac{r_{sk}}{4\,G_N^{(3)}}\,ln\left[e^{2\rho_0}\,sin^2\left(\frac{\pi l}{L}\right)\right] = \frac{r_{sk}}{2\,G_N^{(3)}}\,ln\left[e^{\rho_0}\,sin\frac{\pi l}{L}\right] \tag{6.16}$$

where we have used the common logarithm power rule B.4 to simplify the result. We now recall the relationship between the cutoffs (6.2) and postulate a numerical factor of π: $e^{\rho_0} = L/(\pi a)$.[8] We now see that this result matches the conformal field theory result (4.18) under the condition:

$$c = \frac{3r_{sk}}{2G_N^{(3)}} \tag{6.17}$$

where c represents the central charge of our CFT.

$$S_A \simeq \frac{c}{3}\,ln\left[\frac{L}{\pi a}\,sin\left(\frac{\pi l}{L}\right)\right] \tag{6.18}$$

■

These results are very significant. They show that the maximum entropy capacity of an interval in CFT with one spatial dimension scales *logarithmically*, not linearly, with the size of the interval. This is a much more restrictive upper bound than one would naturally expect.

[8]The relation (6.2) is a *proportionality*, not an equality. The factor of proportionality depends on the particular system in question.

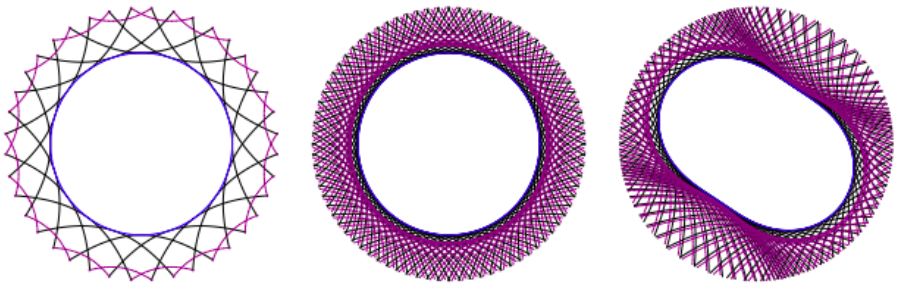

Figure 6.1: A depiction of three bulk curves (in blue) inside a Poincaré disk. The patterns of purple and black geodesics can be used to create a one-to-one correspondence between each bulk curve and a set of (overlapping) intervals on the boundary of the Poincaré disk. In this way, the bulk curves can be described purely in terms of the information content residing on the conformal boundary. [7]

6.3 Holography

The AdS_3/CFT_2 correspondence provides us with a blueprint for describing two-dimensional objects in the bulk Poincaré disk in terms of one-dimensional intervals on the CFT boundary. For any two-dimensional region within the disk, we draw a set of geodesics which are all tangent to its boundary. The set of endpoints of the geodesics correspond to a set of intervals on the CFT space which can be studied using ordinary CFT methods from Chapters 4 and 5. Thus the one-dimensional information on the conformal boundary is sufficient for reconstructing the bulk curve which bounds the two-dimensional region of interest. If we then wish to describe phenomena occurring *within* this bulk region, we simply divide it into sub-regions and execute this process again. There is no limit to how small we can make the sub-regions, and hence we can describe the 2-dimensional phenomena in terms of the 1-dimensional intervals.[9] See figure 6.1.

[9]For more information on the mathematical details, see references [6] and [7].

Chapter 7: Conclusion

In this paper, we have presented a near-comprehensive description of the geometry of anti-de Sitter space and conformal field theory and shown how these seemingly disparate theories are actually related as an example of the holographic principle. The lengths of geodesics (regulated by an ultraviolet cutoff) are directly proportional to the entanglement entropy of an interval whose endpoints on the conformal boundary (CFT_2) match the endpoints of the geodesic in AdS_3. In this way, we can draw a one-to-one correspondence between any two-dimensional region within the bulk AdS_3 with a set of one-dimensional intervals on the conformal boundary simply by drawing all the geodesics which are tangent to the boundary of the two-dimensional region and demarcating their endpoints. This allows us to describe the entire physics of the curved $(2+1)$-dimensional bulk spacetime in terms of physics on the flat $(1+1)$-dimensional conformal boundary, suggesting that the $(2+1)$-dimensional spacetime is a mere "holographic projection" of the information stored on the $(1+1)$-dimensional spacetime surface.

We have also presented original MATLAB code for making approximate calculations to the entanglement entropy of a free, massless, scalar quantum field in a one-dimensional interval and shown how those results are consistent with the holographic principle. Though this code has been designed with this specific system in mind, it would not at all be difficult to modify this algorithm to calculate the entanglement entropy of a system of one's choosing – all one needs to do is determine the appropriate K matrix for the system of interest.

The holographic principle is likely to be an important concept in physics moving forward, as it provides us with a window into understanding theories of quantum gravity in terms of quantum field theories which are already well-understood. (And of course, finding a correct theory of quantum gravity has been extremely challenging.) This principle also may help us better understand the arrow of time, the observed increase of the entropy (information content) of the universe at later times.[1] Such a profound and surprising principle cries

[1] This is a particularly intriguing question in physics because almost all physical phenomena are time reversal symmetric, making it difficult to understand how an asymmetry of time

out for an explanation, likely in the number of degrees of freedom present in a unified theory of spacetime and matter.

It is also conceivable that the holographic principle could have practical benefits regarding information storage, potentially allowing engineers to dramatically reduce the size of information storage chips and devices. The universe is remarkably efficient at storing information, so can humanity tap into this potential? Only time will tell.

could emerge when studying macroscopic phenomena. It is the author's belief that resolving the tension between time reversal symmetry and the arrow of time may require an improved understanding of the quantum measurement problem and possibly a reinterpretation of the insights on the nature of time provided by the theory of relativity.

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Appendix A: General Relativity

$$ds^2 = g_{ab}dx^a dx^b \quad \text{(spacetime interval \& metric tensor)} \tag{A.1}$$

$$g_{ab}g^{bc} = \delta^c_a \quad \text{(inverse metric tensor)} \tag{A.2}$$

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}) \quad \text{(Christoffel symbols)} \tag{A.3}$$

$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed} \quad \text{(Riemann curvature tensor)} \tag{A.4}$$

$$R_{ab} = R^c_{acb} \quad \text{(Ricci Tensor)} \tag{A.5}$$

$$R = g^{ab}R_{ab} \quad \text{(Ricci scalar/scalar curvature)} \tag{A.6}$$

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R \quad \text{(Einstein tensor)} \tag{A.7}$$

$$G_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4}T_{ab} \quad \text{(Einstein field equations)} \tag{A.8}$$

In this study, we restrict ourselves to cases where $T_{ab} \equiv 0$. This corresponds to models in which there is no matter, energy, or momentum present, with the exception of energy due to the cosmological term.

$$\frac{d^2x^a}{d\lambda^2} + \Gamma^a_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0 \quad \text{(geodesic equation)} \tag{A.9}$$

where λ is an affine parameter.

Appendix B: Complex Numbers and Transcendental Functions

B.1 Complex Numbers

Any complex number z can be written in two forms:

$$z = a + bi \,\big|\, a, b \in \mathbb{R} \quad \text{(Cartesian form)} \tag{B.1a}$$

$$z = re^{i\theta} \,\big|\, r > 0,\ 0 \leq \theta < 2\pi \quad \text{(polar form)} \tag{B.1b}$$

These two forms are related by the transformation:

$$\begin{cases} a^2 + b^2 = r^2 & \text{(B.2a)} \\ \dfrac{b}{a} = tan\,\theta & \text{(B.2b)} \end{cases}$$

B.2 Logarithms & Exponentials

$$log(xy) = log\,x + log\,y \quad \text{(Multiplication rule)} \tag{B.3}$$

$$log(x^n) = n\,log\,x \quad \text{(Power rule)} \tag{B.4}$$

ln denotes a logarithm of base e and is called the natural logarithm. To take the natural logarithm of a complex number z, write it in polar form. Then:

$$ln(re^{i\theta}) = ln\,r + ln(e^{i\theta}) = ln\,r + i\theta \tag{B.5}$$

B.3 Trigonometric Functions

B.3.1 Definitions

Circular

Given periodic (circular) trigonometric functions $sin\,x$ and $cos\,x$, we define:

$$tan\,x = \frac{sin\,x}{cos\,x} \tag{B.6a}$$

$$cot\,x = \frac{cos\,x}{sin\,x} \tag{B.6b}$$

$$sec\,x = \frac{1}{cos\,x} \tag{B.6c}$$

$$csc\,x = \frac{1}{sin\,x} \tag{B.6d}$$

Hyperbolic

And analogously for the hyperbolic trigonometric functions $sinh\,x$ and $cosh\,x$:

$$tanh\,x = \frac{sinh\,x}{cosh\,x} \tag{B.7a}$$

$$coth\,x = \frac{cosh\,x}{sinh\,x} \tag{B.7b}$$

$$sech\,x = \frac{1}{cosh\,x} \tag{B.7c}$$

$$csch\,x = \frac{1}{sinh\,x} \tag{B.7d}$$

B.3.2 Identities

Circular

$$sin^2x + cos^2x = 1 \tag{B.8a}$$

$$tan^2x + 1 = sec^2x \tag{B.8b}$$

$$cot^2x + 1 = csc^2x \tag{B.8c}$$

Hyperbolic

$$cosh^2 x - sinh^2 x = 1 \tag{B.9a}$$
$$1 - tanh^2 x = sech^2 x \tag{B.9b}$$
$$cot^2 x - 1 = csch^2 x \tag{B.9c}$$

B.3.3 Relationships with Exponential and Logarithmic Functions

Euler's famous formula $e^{i\pi} + 1 = 0$ and the Taylor series associated with the trigonometric functions (both circular and hyperbolic) establish a link between exponential and trigonometric functions as follows:

$$cos\, x = \frac{e^{ix} + e^{-ix}}{2} \tag{B.10a}$$
$$cosh\, x = \frac{e^{x} + e^{-x}}{2} \tag{B.10b}$$
$$sin\, x = \frac{e^{ix} - e^{-ix}}{2i} \tag{B.10c}$$
$$sinh\, x = \frac{e^{x} - e^{-x}}{2} \tag{B.10d}$$

From these formulas, it is clear how to interpret an imaginary argument in any trigonometric function. We can easily derive the relations:

$$cos(i\theta) = cosh\, \theta \tag{B.11a}$$
$$cosh(i\theta) = cos\, \theta \tag{B.11b}$$
$$sin(i\theta) = -i\, sinh\, \theta \tag{B.11c}$$
$$sinh(i\theta) = -i\, sin\, \theta \tag{B.11d}$$

We will also make use of the trigonometric addition formulas:

$$cos(a + b) = cos\, a\, cos\, b - sin\, a\, sin\, b \tag{B.12a}$$
$$sin(a + b) = sin\, a\, cos\, b + cos\, a\, sin\, b \tag{B.12b}$$

A combination of the above formulas tells us how we should evaluate a trigonometric function with a complex argument. Given a complex number z, write

it as: $z = a + bi$. Then:

$$\begin{aligned} cos(a+bi) &= cos\,a\,cosh\,b \;+\; i\,sin\,a\,sinh\,b \\ sin(a+bi) &= sin\,a\,cosh\,b \;-\; i\,cos\,a\,sinh\,b \end{aligned} \tag{B.13}$$

B.3.4 A Few Short Theorems

Theorem B.3.1. *For large x, we can approximate $x \approx ln\,[2\,cosh\,x] \approx ln\,[2\,sinh\,x]$.*

Proof. We write the hyperbolic cosine and sine functions in exponential form:

$$cosh\,x = \frac{e^x + e^{-x}}{2} \tag{B.14a}$$

$$sinh\,x = \frac{e^x - e^{-x}}{2} \tag{B.14b}$$

For large x, $e^x \gg e^{-x}$. So we can approximate:

$$cosh\,x \approx \frac{e^x}{2} \Rightarrow e^x \approx 2\,cosh\,x \Rightarrow x \approx ln\,[2\,cosh\,x] \tag{B.15a}$$

$$sinh\,x \approx \frac{e^x}{2} \Rightarrow e^x \approx 2\,sinh\,x \Rightarrow x \approx ln\,[2\,sinh\,x] \tag{B.15b}$$

■

Theorem B.3.2.

$$ln\left(\frac{1+p}{1-p}\right) = 2\,tanh^{-1}(p) \tag{B.16}$$

Proof.

$$tanh\,b = \frac{sinh\,b}{cosh\,b} = \frac{(e^b - e^{-b})/2}{(e^b + e^{-b})/2} = \frac{(e^b - e^{-b})}{(e^b + e^{-b})} = \frac{(e^{2b} - 1)}{(e^{2b} + 1)} \tag{B.17}$$

$$\begin{aligned} \Rightarrow\; tanh\left[\frac{1}{2}\,ln\left(\frac{1+p}{1-p}\right)\right] &= \frac{(e^{2\cdot\frac{1}{2}\,ln\left(\frac{1+p}{1-p}\right)} - 1)}{(e^{2\cdot\frac{1}{2}\,ln\left(\frac{1+p}{1-p}\right)} + 1)} = \frac{(e^{ln\left(\frac{1+p}{1-p}\right)} - \frac{1-p}{1-p})}{(e^{ln\left(\frac{1+p}{1-p}\right)} + \frac{1-p}{1-p})} = \\ &= \frac{\frac{1+p}{1-p} - \frac{1-p}{1-p}}{\frac{1+p}{1-p} + \frac{1-p}{1-p}} = \frac{(1+p) - (1-p)}{(1+p) + (1-p)} = \frac{2p}{2} = p \end{aligned} \tag{B.18}$$

Therefore:

$$ln\left(\frac{1+p}{1-p}\right) = 2\,tanh^{-1}(p) \tag{B.19}$$

■

B.3.5 Double-Angle and Half-Angle Formulas

Given (B.12), we can derive the following double-angle formulas:

$$sin(2x) = 2\,sin\,x\,cos\,x \tag{B.20a}$$

$$cos(2x) = cos^2\,x - sin^2\,x = 1 - 2sin^2\,x = 2cos^2\,x - 1 \tag{B.20b}$$

$$tan(2x) = \frac{2\,tan\,x}{1 - tan^2\,x} \tag{B.20c}$$

We can use these formulas to calculate the half-angle formulas:

$$sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - cos\,x}{2}} \tag{B.21a}$$

$$cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 + cos\,x}{2}} \tag{B.21b}$$

$$tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - cos\,x}{1 + cos\,x}} = \frac{sin\,x}{cos\,x + 1} = \frac{cos\,x - 1}{sin\,x} \tag{B.21c}$$

B.3.6 Reduction Formulas

$$sin(-x) = -sin\,x \tag{B.22a}$$

$$cos(-x) = cos\,x \tag{B.22b}$$

$$sin(\pi - x) = sin\,x \tag{B.22c}$$

$$cos(\pi - x) = -cos\,x \tag{B.22d}$$

B.3.7 Taylor Series and Small-Angle Approximations

$$sin\,x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \tag{B.23a}$$

$$\Rightarrow\ sin\,x \approx x \quad \text{for small x} \tag{B.23b}$$

$$cos\,x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \tag{B.23c}$$

$$\Rightarrow\ cos\,x \approx 1 - \frac{x^2}{2} \quad \text{for small x} \tag{B.23d}$$

Appendix C: Linear Algebra

Given λ_a is an eigenvalue of matrix A and k is any positive integer:

$$tr(A) = \sum \text{diagonal elements of } A = \sum_a \lambda_a \tag{C.1}$$

$$tr(A^k) = \sum_a \lambda_a^k \tag{C.2}$$

For two matrices A and B with dimensions $m \times n$ and $n \times m$, respectively:

$$tr(AB) = tr(BA) \tag{C.3}$$

For any matrix M, we denote its conjugate by M^* (which is obtained by replacing each element with its own complex conjugate), its transpose by M^T, and its conjugate transpose by $M^\dagger$.

$$(M^*)^* = M, \quad (M^T)^T = M, \quad (M^\dagger)^\dagger = M \tag{C.4}$$

For any two compatible matrices C and D, the regular rule of complex conjugate multiplication holds:

$$(CD)^* = C^* D^* \tag{C.5}$$

The transpose of C and D is given by:

$$(CD)^T = D^T C^T \tag{C.6}$$

And their conjugate transpose is given by:

$$(CD)^\dagger = D^\dagger C^\dagger \tag{C.7}$$

A Hermitian matrix E is one which is equal to its own conjugate transpose:

$$E = E^\dagger \tag{C.8}$$

Eigenvalues of Hermitian matrices are always real numbers. Mathematically, if λ_i denotes an eigenvalue of E, then:

$$\lambda_i \in \mathbb{R},\ \forall\ i \tag{C.9}$$

Let F be some arbitrary matrix and let $G = FF^\dagger$. Then G is Hermitian:

$$G^\dagger = (FF^\dagger)^\dagger = (F^\dagger)^\dagger F^\dagger = FF^\dagger = G \tag{C.10}$$

If a matrix G is Hermitian, then it has the same set of eigenvalues λ_i^G as its conjugate G^*, $\lambda_i^{G^*}$:

$$\{\lambda_i^G\} = \{\lambda_i^{G^*}\} \tag{C.11}$$

The reason for this is as follows: Since G is Hermitian, there must exist some basis in which it can be expressed in a diagonal form. In this basis, all matrix elements must be real numbers since the diagonal elements are the eigenvalues, which are all real for a Hermitian matrix, and the off-diagonal elements are all zero. Therefore $G^* = G$ in this basis. But G^* is also Hermitian: $G^* = (G^\dagger)^T = (G)^T = (G^*)^\dagger$, so it too must be diagonalizable. But we have already found a basis in which G^* can be expressed in diagonal form – the same basis we used to diagonalize G. In this unique basis, the eigenvalues for G and G^* can simply be read off the diagonal of this matrix, and since G and G^* are identical in this basis, their sets of eigenvalues must also be identical.

The results of linear algebra can be applied to more general situations. For example, our "vectors" can be thought of as functions, and the outer product of two vectors could be thought of as a multivariable function which is the product of two single-variable functions:

$$h(x, y) = f(x)g(y) \tag{C.12}$$

This multivariable function can now be treated like a matrix with an uncountably infinite number of entries. The trace of this multivariable function is given by setting the arguments x and y equal and then integrating the function:

$$tr\,[h(x, y)] = \int_{-\infty}^{\infty} h(x, x)\,dx = \int_{-\infty}^{\infty} f(x)\,g(x)\,dx \tag{C.13}$$

For a function with more than two entries, one can perform a partial trace by only tracing over a subset of the variables. (In general, to trace over a pair of variables, one sets those two variables equal to one another and then integrates over all possible values for that common variable.)

C.1 Diagonalizing a Matrix

A square matrix Q is said to be *orthogonal* if its rows and columns are orthonormal vectors. A necessary and sufficient condition for a square matrix Q to be orthogonal is that its transpose and its inverse are equivalent:

$$Q^T = Q^{-1} \tag{C.14}$$

A square matrix H is said to be *diagonalizable* if there exists an invertible matrix P such that:

$$P^{-1}HP = H_D \tag{C.15}$$

where H_D is a diagonal matrix. H_D can be thought of as H expressed in a different basis. If H happens to be real and symmetric, then its eigenvectors are orthonormal and so the change of basis matrix P will be orthogonal.

Appendix D: Miscellaneous

D.1 Hermite Polynomials

The family of Hermite polynomials $H_n(x)$ are each a general solution to the differential equation:

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0 \tag{D.1}$$

for non-negative integer n. The Hermite polynomials are given by the Rodrigues formula:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) = \left(2x - \frac{d}{dx}\right)^n \cdot 1 \tag{D.2}$$

where $\left(2x - \frac{d}{dx}\right)$ is thought of an operator which, once raised to the n^{th} power, operates on the constant function 1.

D.2 Infinite Series

Given $|x| < 1$, the sum $\sum_{k=0}^{\infty} x^k$ is convergent and is given by:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x} \tag{D.3}$$

One can differentiate both sides of this equation to obtain:

$$\sum_{k=0}^{\infty} kx^{k-1} = 0 + 1 + 2x + 3x^2 + \ldots = \frac{1}{(1-x)^2} \tag{D.4}$$

We can now start the sum at $k = 1$, as the $k = 0$ term will not contribute. We can also multiply both sides of this equation by x to obtain a key formula:

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{x}{(1-x)^2} \tag{D.5}$$

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