

DEPARTMENT OF PHYSICS
BROWN UNIVERSITY
Written Qualifying Examination for the Ph.D. Degree
January, 2019

READ THESE INSTRUCTIONS CAREFULLY

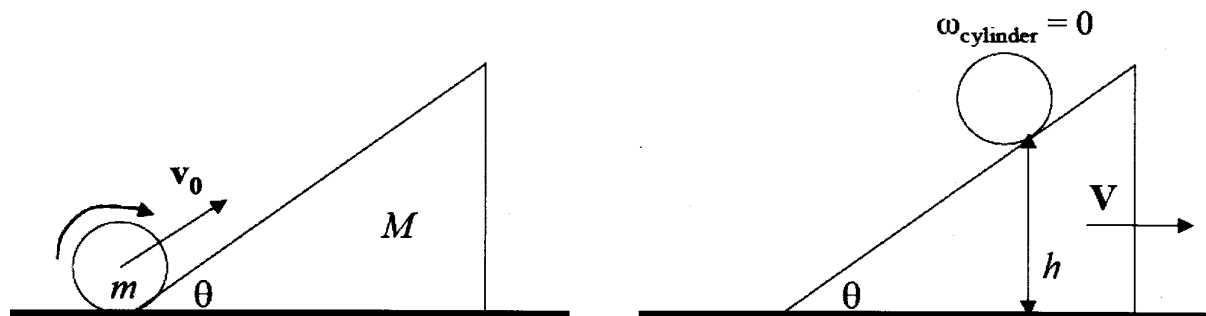
1. The time allowed to complete the exam is 10:00–3:00 PM.
2. All work is to be done without the use of books or papers and without help from anyone. The use of calculators or other electronic devices is also not permitted.
3. Use a separate answer book for each question, or two books if necessary.
4. **DO NOT write your name in your booklets.** Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.
5. Write the problem number in the center of the outside front cover. **Write nothing else** on the inside or outside of the front and back covers. Note that there are separate graders for each question.
6. Answer one (and **only** one) problem from each of the five pairs of questions. The pairs are labeled as follows:

Classical Mechanics	CM1, CM2
Electricity and Magnetism	EM1, EM2
Statistical Mechanics	SM1, SM2
Quantum Mechanics	QM1, QM2
Quantum Mechanics	QM3, QM4

Note that there are two pairs of Quantum Mechanics problems. You have to do **one** problem from **each** pair.

7. All problems have equal weight.

1. CM-1



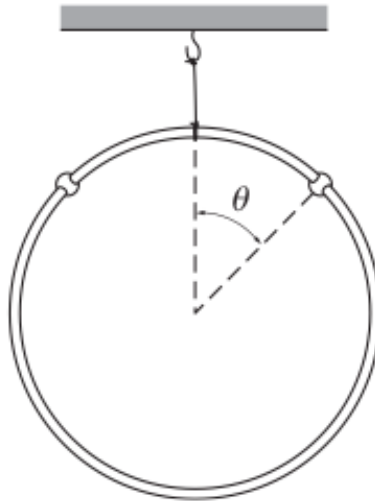
A hollow cylinder of radius R and mass m rolls up an inclined plane of angle θ without slipping. The inclined plane has mass M and is free to slide along the horizontal surface without friction. The cylinder has an initial velocity \vec{v}_0 up the inclined plane. Furthermore, the inclined plane is initially at rest with respect to the horizontal surface.

(a) [4 points] After some time the cylinder stops rotating and begins to roll back down the inclined plane. At this moment, what is the horizontal component of velocity, denoted V , of the cylinder and inclined plane?

(b) [6 points] How high has the cylinder risen at this point? (This is the vertical distance to the **bottom** of the cylinder, marked h in the figure).

Please give your answers in terms of m, M, θ, v_0 and the gravitational constant g .

2. CM-2



Two beads, each of mass m , can move without friction around a circular wire ring. The ring, which has mass M and radius R , is suspended from the ceiling by a massless string.

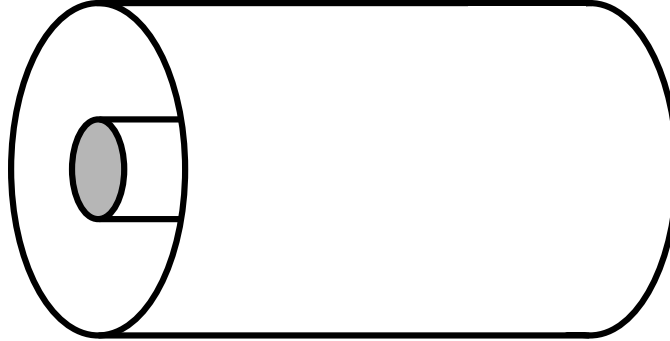
The two beads are held at the top of the ring and then released simultaneously so that they fall symmetrically, one down the left side and the second down the right. Remarkably, under certain conditions, the ring appears to jump up once the beads have fallen sufficiently.

(a) [7 points] Calculate the angle θ from the vertical at which the beads are positioned when the tension in the string first goes to zero.

(b) [3 points] What condition must the ratio M/m of the masses satisfy in order for the tension in the string to go to zero?

3. EM-1

Shown below is a portion of coaxial cable of length ℓ , consisting of a conducting solid inner cylinder of radius a and a conducting cylindrical outer shell of radius b , separated from each other by a nonconducting material which you can assume to be vacuum. You may assume the cable is very long ($\ell \gg b > a$) so that ‘edge effects’ can be ignored.



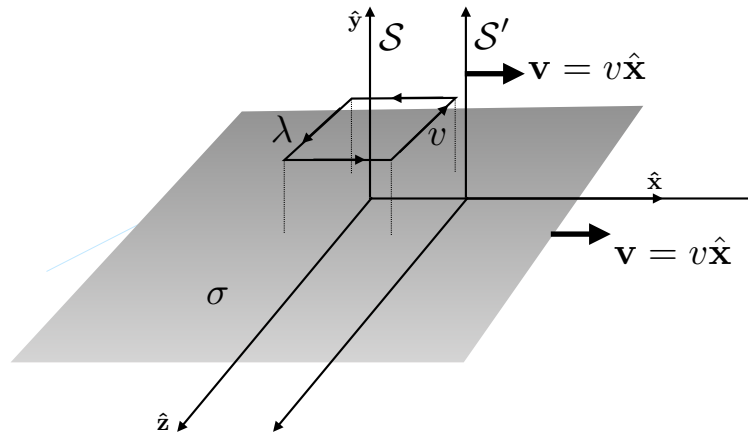
If the inner cylinder has a uniform static charge density of λ Coulombs/meter along the wire, and the outer shell has a uniform static charge density of $-\lambda$ Coulombs/meter, then:

- (a) [2 points] determine the electric field everywhere (no credit for a memorized answer, you must explain your reasoning), and
- (b) [2 points] calculate the total electrostatic energy stored in the cable.

Now, if instead of the previous scenario, suppose that the inner cylinder and the outer shell are both electrically neutral but instead they carry currents, each of magnitude I amperes, flowing in opposite directions (the current on the inner cylinder flows to the right, that on the outer shell flows to the left). Then:

- (c) [3 points] determine the magnetic field everywhere inside the cable (no credit for a memorized answer, you must explain your reasoning; also, be sure to indicate the direction of the magnetic field by sketching the field lines as they would appear if you looked along the axis of the cable from the left),
- (d) [2 points] calculate the total magnetic energy stored in the cable, and
- (e) [1 point] calculate the self-inductance of the cable.

4. EM-2



In the frame \mathcal{S} , a square loop of wire is situated at rest some distance above an infinite uniform surface charge density σ . The surface charge σ moves to the right along the x -axis with velocity $\vec{v} = v\hat{x}$. The sides of the loop have length ℓ , and the loop carries a closed current in the form of a uniform line charge density λ which flows around the loop with speed v (in the direction indicated in the figure). Do not assume that $v \ll c$.

- (a) [1 point] What is the magnetic dipole moment of the wire loop in frame \mathcal{S} ?
- (b) [3 points] Find the torque on the wire loop in frame \mathcal{S} .

Frame \mathcal{S}' is the rest frame of the surface charge; it moves along the x direction with speed v relative to frame \mathcal{S} , as shown in the figure.

- (c) [3 points] What is the electric dipole moment of the wire loop in frame \mathcal{S}' ?
- (d) [3 points] Find the torque on the wire loop in frame \mathcal{S}' , and show that it agrees with the result of part (b) up to an overall factor of γ .

5. SM-1

Helium atoms (of mass m) can be adsorbed onto the surface of a metal, and an amount of work ϕ is then necessary to remove a helium atom from the metal surface to infinity. The helium atoms are completely free to move, without mutual interaction, on the 2-dimensional metal surface. Suppose such a metal surface of area A is in contact with with helium gas of volume V at pressure P , and the whole system is in equilibrium at temperature T .

(a) [3 points] What is the free energy of the gas, in terms of N_g (the number of indistinguishable molecules of gas), quantities given in the problem, and fundamental constants?

(b) [3 points] What is the free energy of the adsorbate, in terms of N_s (the number of indistinguishable molecules adsorbed on the surface), quantities given in the problem, and fundamental constants?

(c) [4 points] Show that the equilibrium condition is

$$\frac{\partial F}{\partial N_g} = \frac{\partial F}{\partial N_s}$$

where F is the total free energy, and use this relation to calculate the mean number of atoms adsorbed per unit area on the metal surface.

Possibly useful formulas:

$$\ln N! \approx N \ln N - N$$
$$\int_0^\infty e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}}$$

6. SM-2

A long and wiggly polymer is often modeled as a freely-jointed chain of a large number N of short segments of length L_{seg} . Although neighboring segments are joined, they can rotate freely with respect to each other. Let us simplify the problem to one dimension, by assuming that each segment is aligned in either the $+x$ or $-x$ direction. So, the position of the end of segment i relative to that of the preceding segment is $L_{\text{seg}}\sigma_i$, where $\sigma_i = \pm 1$ indicates that segment i is oriented in the $\pm x$ direction. The total end-to-end displacement of the polymer chain can be expressed as

$$x = L_{\text{seg}} \sum_{i=1}^N \sigma_i.$$

Suppose one grabs both ends of a long chain as described, pulling them apart with a force f , and measuring the average end-to-end distance of the chain as a function of the force.

(a) [7 points] Construct a partition function for the system based on the expression of probability of a given configuration $\{\sigma_1, \sigma_2, \dots, \sigma_N\}$ to calculate $\langle x \rangle$ as a function of f . Here the average $\langle x \rangle$ is taken over all configurations. The result is known as the force-extension curve for a freely-jointed chain.

(b) [3 points] Simplify the result at the high force and low force limits, and comment briefly what these two answers physically mean.

7. QM-1

Consider an interacting electron-positron system whose Hamiltonian is

$$H = A (\mathbf{S}^{(e-)} \cdot \mathbf{S}^{(e+)}) + \frac{eB}{mc} (S_z^{(e-)} - S_z^{(e+)}),$$

where A and B are constants, m is the mass of the electron, and $\mathbf{S}^{(e-)}$ and $\mathbf{S}^{(e+)}$ are the spin operators for the electron and positron respectively. These are related to the Pauli matrices by

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Take a spin wavefunction given by $\psi = \chi_+^{(e-)} \chi_-^{(e+)}$ where

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(a) [4 points] Show that in the limit $A \rightarrow 0$, ψ is an eigenstate of H , and compute its eigenvalue.

(b) [6 points] Now take the limit $B \rightarrow 0$, $A \neq 0$. Is ψ still an eigenstate of H ? Find the expectation value of H in this limit (as a function of A).

8. QM-2

A particle of mass m is contained within an impenetrable one-dimensional well extending from $x = -\frac{L}{2}$ to $x = +\frac{L}{2}$. The particle is in its ground state.

(a) [2 points] Find the eigenfunctions of the ground state and the first excited state.

(b) [6 points] The walls of the well are instantaneously moved outward to form a new well extending from $-L < x < +L$. Calculate the probability that the particle will stay in the ground state (of the new well configuration) during this sudden expansion.

(c) [2 points] Calculate the probability that the particle jumps from the initial ground state to the first excited final state of the expanded well.

9. QM-3

A particle of mass m moves in one dimension. The Hamiltonian

$$\hat{H} = \frac{(\hat{p} + \alpha\hat{x})^2}{2m},$$

where α is a real constant.

(a) [2 points] The form of the given Hamiltonian is that of a charged particle moving in the presence of a magnetic vector potential A . For a particle of charge q , for what value of A would the Hamiltonian take the indicated form \hat{H} ?

(b) [3 points] Perform a gauge transformation to eliminate A ; indicate how the required transformation acts on wave functions of the charged particle, and give the new, gauge-transformed Hamiltonian \hat{H}' .

(c) [5 points] Find all energy levels and a complete set of eigenfunctions of the original Hamiltonian \hat{H} .

10. QM-4

Consider a particle of charge e and mass m in constant electric and magnetic fields

$$\vec{\mathbf{E}} = \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix}, \quad \vec{\mathbf{B}} = \begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix}.$$

(a) [3 points] Write the Schrödinger equation in the gauge

$$\vec{\mathbf{A}} = \begin{pmatrix} Bz \\ 0 \\ 0 \end{pmatrix}, \quad \phi = -Ez.$$

(b) [3 points] Separate variables and reduce it to a one-dimensional problem.

(c) [4 points] Find the expectation value of the velocity in the x direction in any eigenstate (the drift velocity).