DEPARTMENT OF PHYSICS
BROWN UNIVERSITY
Written Qualifying Examination for the Ph.D. Degree
January, 2018

READ THESE INSTRUCTIONS CAREFULLY

1. The time allowed to complete the exam is 10:00–3:00 PM.
2. All work is to be done without the use of books or papers and without help from anyone. The use of calculators or other electronic devices is also not permitted.
3. Use a separate answer book for each question, or two books if necessary.
4. **DO NOT write your name in your booklets.** Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.
5. Write the problem number in the center of the outside front cover. **Write nothing else** on the inside or outside of the front and back covers. Note that there are separate graders for each question.
6. Answer one (and **only** one) problem from each of the five pairs of questions. The pairs are labeled as follows:

Classical Mechanics	CM1, CM2
Electricity and Magnetism	EM1, EM2
Statistical Mechanics	SM1, SM2
Quantum Mechanics	QM1, QM2
Quantum Mechanics	QM3, QM4

   Note that there are two pairs of Quantum Mechanics problems. You have to do **one** problem from **each** pair.
7. All problems have equal weight.

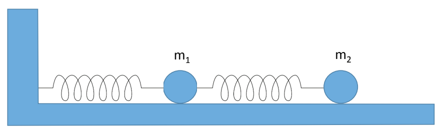
1. CM-1

Consider two masses $m_1 = m_2$ connected by a spring with spring constant k. Mass m_1 is connected to a rigid wall by another spring also of spring constant k. The two masses are on a horizontal frictionless table.

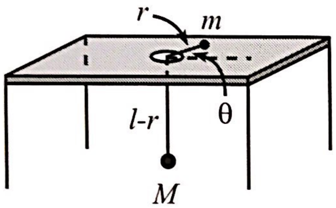
(a) Write down the Lagrangian of the problem using the masses' displacements x_1 and x_2 from their equilibrium positions;

(b) Derive the equations of motion of the two masses;

(c) Find the resonance frequencies and the ratios of x_1/x_2 for the normal modes.



2. CM-2



A mass, m, is free to slide on a frictionless table. It is connected to a string of total length, l, that passes through a hole to a mass, M, that hangs a distance $l - r$ below. Assume that the string is always taut and M only moves vertically.

(a) Find the equations of motion for the variables r and θ.

(b) Under what condition does m undergo circular motion?

(c) What is the frequency of small oscillations of the variable r about the trajectory you found in part (b)?

(d) If the relative amplitude of the radial oscillations is $a = \delta r / r_0$, where r_0 is the average distance of mass m from the hole, then what is the relative amplitude of the oscillations in the angular frequency of the circular motion?

3. EM-1

A particle of mass m and charge e moves at a constant, non-relativistic speed $v \ll c$ in a circle with radius l.
The electric radiation field at a large distance r from the charge is

$$\mathbf{E}_{\text{rad}} = \frac{e}{4\pi\epsilon_0 c^2 r}\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{a}),$$

where $\mathbf{a}$ is the acceleration of the charge, and $\hat{\mathbf{n}}$ is the unit vector connecting the charge e and the observer.

(a) Find the amplitude of the oscillating electric field as a function of r and the angle θ with respect to the axis of rotation of the particle.

(b) What is the average power radiated per unit solid angle in direction θ?

(c) What is the polarization of the radiation if $\theta = 0$? What is the polarization if $\theta = \pi/2$?

(d) What is the frequency of the emitted radiation?

4. EM-2

(a) Show that in the presence of time-independent electric and magnetic fields, the general equation of motion for a non-relativistic particle of mass m and charge q can be written as

$$\frac{d^2\mathbf{v}}{dt^2} = -\omega_c^2\mathbf{v} + \frac{q^2}{m^2}\left[(\mathbf{E}\times\mathbf{B}) + (\mathbf{B}\cdot\mathbf{v})\mathbf{B}\right],$$

where $\omega_c = qB/m$ is the cyclotron frequency (note that the equation of motion depends only on the charge-to-mass ratio).

(b) Now, let $\mathbf{E} = E_0\hat{\mathbf{y}}$, and $\mathbf{B} = B_0\hat{\mathbf{z}}$. Assuming an initial condition such that the particle has no velocity along $\hat{\mathbf{z}}$ at $t = 0$, show that

$$\mathbf{v}(t) = v_0\cos(\omega_c t + \phi)\hat{\mathbf{x}} + v_0\sin(\omega_c t + \phi)\hat{\mathbf{y}} + \frac{\mathbf{E}\times\mathbf{B}}{B^2},$$

where v_0 and ϕ are constants that depend on the initial conditions.

(c) Sketch a trajectory in the $x - y$ plane.

5. SM-1

A protein can be modelled as a system of N particles. The protein has one folded state of the total energy $-\epsilon_b$ and μ^N unfolded states of 0 energy, where μ is a constant. The fraction of protein molecules that are in unfolded states at temperature T is $P_u(T)$.

(a) Write down the partition function for the protein in terms of ϵ_b, N, μ, and T.

(b) Find the unfolding transition temperature T_t at which half of the proteins are in unfolded states.

(c) Show that the unfolding transition becomes very sharp as N increases by computing the width of the unfolding transition, $\delta T_t = \left(\frac{dP_u}{dT}\right)^{-1}$, evaluated at T_t.

6. SM-2

Consider a classical one dimensional anharmonic oscillator. The potential energy is

$$u(x) = cx^2 - gx^3 + fx^4,$$

where the anharmonic coefficients g and f are positive and small.

(a) Show that the classical partition function can be written in the leading orders in f and g as

$$Z = TB\left(1 + AT\right),$$

where A is a constant depending on c, f, and g, and B does not depend on f and g.

(b) Show that to two leading orders in T at low temperature the classical specific heat has the form

$$C_v \approx k_B + 2Ak_BT.$$

Show that to leading order in T the average coordinate $\langle x \rangle$ is given by

$$\langle x \rangle \approx \frac{3}{4}\frac{gk_BT}{c^2}.$$

You may find the following useful:

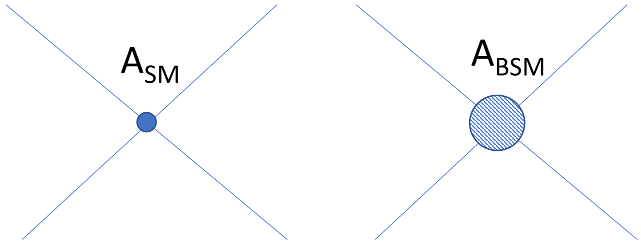
$$\ln\left(1 + x\right) \approx x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

For an even n,

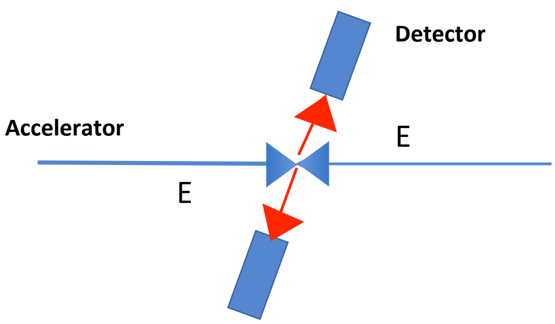
$$\int_{-\infty}^{\infty} x^n e^{\frac{-x^2}{2\sigma^2}} dx = \sqrt{2\pi}\,(n-1)!!\sigma^{(n+1)},$$

where $m!!$ is the double factorial equal to the product of odd integers from 1 to m for m odd, and to the product of even integers from 2 to m for m even. Also note that $-1!! = 0!! = 1$.

7. QM-1



You are designing an experiment to study quantum interference of two scattering processes, shown in a figure above. The first one is a standard model (SM) scattering with an amplitude $A_{\text{SM}} \sim \exp(-M/M_0)$, where M is the sum of the energies of the two scattered particles and M_0 is the characteristic energy scale for the SM scattering. The other is a beyond SM (BSM) process, not quite known, so we parameterize it via an effective interaction with an amplitude $A_{\text{BSM}} = A_{\text{SM}} \times f \times \left(\frac{M}{4\pi\Lambda}\right)^2$. Here, $\Lambda \gg M_0$ is an unknown parameter governing the BSM interaction and $f = \pm 1$ is the sign of the interference between the SM and BSM processes. This parameterization, known as "effective field theory" approximation, holds for $M \ll \Lambda$.



Your experimental setup consists of an accelerator and a fixed detector (see figure). The accelerator allows you to produce two beams of particles with the same energy E and collide them head-on. The intensity of the beams does not depend on E. Your detector is very simple and only allows you to count the number of events with the outgoing particles scattered in a certain solid angle.

What energy would you choose in order to establish the sign of the interference (i.e., the sign of f) with maximum confidence given the allocated amount of accelerator time?

8. QM-2

A quantum-mechanical system in the absence of perturbations can exist in either of the two states $|1\rangle$ and $|2\rangle$ with energies $E_1 = E_2$. Suppose that it is acted upon by a time-independent perturbation

$$V = \begin{pmatrix} 0 & V_{12} \\ V_{21} & 0 \end{pmatrix},$$

where $V_{21} = V_{12}^*$. If at time $t = 0$, the system is in state $|1\rangle$, determine the amplitude for finding the system in either state at any later time.

9. QM-3

A long dielectric cylinder with length L is uniformly charged throughout its volume. It fills the space between two circular condenser plates, which have a potential difference Φ_0 across them. An electron is free to move in a small canal drilled in the dielectric through its symmetry axis. The effective Hamiltonian of the one-dimensional motion along the canal for this electron is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{Kx^2}{2} + \frac{e\Phi_0}{L}x$$

where x is the location of the electron measured from the center of the cylindrical dielectric block. Assume that the electron wave function is not negligible only far away from the plates.

(a) What are the eigenenergies of $\hat{H}$?

(b) Find the ground state wave function.

10. QM-4

A particle of mass m is in a one-dimensional box with infinite walls at $x = -L/2$ and $x = +L/2$. The box then instantaneously expands so that its walls are now at $x = -L$ and $x = +L$.

(a) Find the eigenfunctions of the ground and first excited states of the particle for the box in its initial configuration.

(b) Find the eigenfunctions of the ground and first excited states of the particle after the box has expanded.

(c) Find the probability that the particle, initially in the ground state, ends up in the first excited state after the box expands.