DEPARTMENT OF PHYSICS
BROWN UNIVERSITY
Written Qualifying Examination for the Ph.D. Degree
January, 2017

READ THESE INSTRUCTIONS CAREFULLY

12-5:00

1. The time allowed to complete the exam is ~~10:00-3:00~~ PM.

2. All work is to be done without the use of books or papers and without help from anyone. The use of calculators or other electronic devices is also not permitted.

3. Use a separate answer book for each question, or two books if necessary.

4. **DO NOT write your name in your booklets.** Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.

5. Write the problem number in the center of the outside front cover. **Write nothing else** on the inside or outside of the front and back covers. Note that there are separate graders for each question.

6. Answer one (and **only** one) problem from each of the five pairs of questions. The pairs are labeled as follows:

Classical Mechanics	CM1, CM2
Electricity and Magnetism	EM1, EM2
Statistical Mechanics	SM1, SM2
Quantum Mechanics	QM1, QM2
Quantum Mechanics	QM3, QM4

   Note that there are two pairs of Quantum Mechanics problems. You have to do **one** problem from **each** pair.

7. All problems have equal weight.

1. CM-1

Consider a pendulum with one end attached to the uppermost point of a fixed disk of radius R. The pendulum is made up of a massless string with a mass m attached to its end as shown in the figure below. A constant gravitational field g is acting downwards in the z-direction. Assume that the total length of the string is l and that $l > \pi R$.
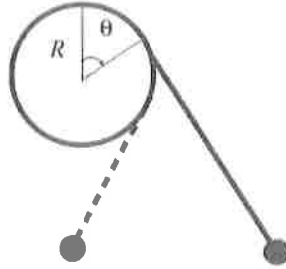
(a) (6 points) Find the equations of motion in terms of the angle θ as shown in the figure.

(b) (1 point) What is the equilibrium angle, θ_0?

(c) (3 points) Find the frequency of oscillations about the equilibrium angle .

2. CM-2

In general relativity, the Lagrangian describing orbits of a test particle around a static point mass may be written in terms of the coordinate four-vector $X^\mu(\tau) = (t(\tau), r(\tau), \theta(\tau), \phi(\tau))$ and the four-velocity $\frac{dX^\mu}{d\tau} = (\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\phi}{d\tau})$, as

$$L\left(X^\mu, \frac{dX^\mu}{d\tau}\right) = \frac{1}{2}\left[(1 - \frac{r_s}{r})c^2(\frac{dt}{d\tau})^2 - \frac{1}{(1 - \frac{r_s}{r})}(\frac{dr}{d\tau})^2 - r^2(\frac{d\theta}{d\tau})^2 - r^2\sin^2(\theta)(\frac{d\phi}{d\tau})^2\right],$$

where τ is the proper time. Consider orbits in the $\theta = \pi/2$ plane (so that $\sin^2(\theta) = 1$ and $\frac{d\theta}{d\tau} = 0$.)

(a) (3 points) Use the Euler-Lagrange equations to show that the quantities:
$r^2\frac{d\phi}{d\tau}$ and $(1 - \frac{r_s}{r})\frac{dt}{d\tau}$ are constants of the motion.

(b) (4 points) Let $\frac{L}{m} \equiv r^2\frac{d\phi}{d\tau}$ and $\frac{E}{mc^2} \equiv (1 - \frac{r_s}{r})\frac{dt}{d\tau}$, and consider the identity (from the definition of the metric tensor:)

$$c^2 = (1 - \frac{r_s}{r})c^2(\frac{dt}{d\tau})^2 - \frac{1}{(1 - \frac{r_s}{r})}(\frac{dr}{d\tau})^2 - r^2(\frac{d\phi}{d\tau})^2$$

Write an expression for $(\frac{dr}{d\tau})^2$ in terms of E, L, and r_s. Use the fact that $r_s = 2GM/c^2$ to rewrite this expression in terms of the kinetic energy, classical gravitational potential energy, total energy, and relativistic corrections. (multiply through by $m/2$). Derive an expression for the effective potential describing the radial motion of a test particle.

(c) (3 points) Sketch the effective potential from part (b) and comment on the existence of circular orbits.

3. EM-1

At time $t = 0$, a particle of mass m and charge q is at rest at the origin in a region filled with a uniform electric field $\vec{E} = E\,\hat{y}$ and a uniform magnetic field $\vec{B} = B\,\hat{z}$.

(a) (6 points) Find the subsequent motion of the particle, assuming $E < cB$. Note that the motion is not necessarily nonrelativistic, so you had better use the relativistic form of the Lorentz force law

$$\frac{dp^\mu}{d\tau} = qu_\nu F^{\mu\nu},$$

where p^μ and u_ν are respectively the momentum and velocity 4-vectors, τ is the proper time, and

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

is the electromagnetic field strength. (You may neglect energy loss due to the radiation of the accelerating particle.)

(b) (4 points) If there are two infinite parallel plates at $y = 0$ and $y = d > 0$, with a potential difference $V = Ed$ between the two plates, and we then apply a magnetic field parallel to the plates, we have a device called a magnetron. If electrons are emitted from rest from one of the plates (called the cathode), how strong must the magnetic field be to prevent them from reaching the other plate (called the anode)?

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4. EM-2

The simplest possible *spherical* electromagnetic wave is described by the equations:

$$\vec{E}(r,\theta,\phi,t) = E_0 \frac{\sin\theta}{r} \left[\cos(kr - \omega t) - \frac{1}{kr} \sin(kr - \omega t) \right] \hat{\phi} \tag{1}$$

$$\vec{B}(r,\theta,\phi,t) = \frac{2 E_0 \cos\theta}{\omega r^2} \left(\sin(kr - \omega t) + \frac{1}{kr} \cos(kr - \omega t) \right) \hat{r}$$

$$+ \frac{E_0 \sin\theta}{\omega r} \left(-k \cos(kr - \omega t) + \frac{1}{kr^2} \cos(kr - \omega t) + \frac{1}{r} \sin(kr - \omega t) \right) \hat{\theta} \tag{2}$$

(a) (6 points) Calculate the time-averaged Poynting vector (often called the "intensity") of this wave. (Hint: let $u \equiv kr - \omega t$, if only to make the equations shorter.)

(b) (2 points) Does the intensity vector point in the expected direction? Does it have the r dependence you would expect?

(c) (2 points) Why is this the simplest spherical electromagnetic wave, i.e., why can't you have a propagating spherical wave with no θ or ϕ dependence?

5. SM-1

Consider a system of 3 classical (Ising) spins $s_1 = \pm 1$, $s_2 = \pm 1$ and $s_3 = \pm 1$ at the vertices of a triangle as shown. The energy of the system is given by

$$H = J\,(s_1\, s_2 + s_2\, s_3 + s_3\, s_1)$$

where $J < 0$.
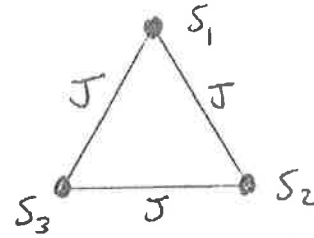
a) (3 points) Find the partition function Z at temperature T.

b) (3 points) Find the free energy F and verify that it has the expected value $(F = E - TS)$ in the limit of small but non-zero temperature.

c) (4 points) Now include a magnetic field B such that $H \to H + B(s_1 + s_2 + s_3)$. Find the magnetization as a function of B and T in the limit $|B| \ll k_b T \ll -J$.

6. SM-2

A graduate student changes the settings of the Large Hadron Collider and accidentally creates a black hole of mass M. The internal energy of the black hole is given by Einstein's relation

$$E = Mc^2.$$

The black hole also has an amount of entropy given by the Hawking-Bekenstein relation

$$S = \frac{k_B c^3}{4Gh} A,$$

where A is the surface area of the black hole whose radius is given by

$$R = \frac{2GM}{c^2}.$$

(Not even light can escape the black hole's gravity from inside R.)

(a) (4 points) Start from the thermodynamic definition of temperature and derive an expression for the black hole's temperature T in terms of its mass M and the fundamental constants G, $\hbar$, c and k_B.

(b) (3 points) A black hole is not truly black; it emits radiation as the result of quantum pair-production processes near its event horizon. Assume the radiated power per unit surface area j obeys the Stefan-Boltzmann law,
$$j = \sigma T^4,$$
where $\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$. Find the rate of energy loss of the black hole.

(c) (3 points) Find the time it takes the black hole to completely evaporate.

Fun bonus: What is the mass of the largest black hole that will evaporate within 1 day, the time it takes the protesters to arrive?

7. QM-1

Consider a spinless particle in three dimensions subject to a harmonic central force, with Hamiltonian

$$H = \frac{\vec{p}^2}{2m} + \frac{1}{2}kr^2$$

(a) (5 points) Find the energy eigenvalues and determine the degeneracy of the lowest four.

(b) (5 points) Now put 5 identical particles into this potential. Assuming the particles do not interact with each other, find the ground state and its energy when:

    (i) they have spin 1/2

    (ii) they have spin 1

    (iii) they have spin 3/2

8. QM-2

An electron is at rest in an oscillating magnetic field given by $\vec{B} = B_0 \cos(\omega t)\hat{k}$, and is initially in the $\frac{\hbar}{2}$ eigenstate of the S_x operator, i.e.,

$$|\psi(0)\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

(a) (5 points) Find $|\psi(t)\rangle$ in terms of B_0, γ, ω and t.

(b) (5 points) Evaluate $\langle \vec{S}(t) \rangle = \langle \psi(t)|\vec{S}|\psi(t)\rangle$. Does your answer for S_z make simple physical sense in terms of the initial state and the applied field?

9. QM-3

A particle of mass m moves in the three-dimensional potential

$$V = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2) + \frac{\lambda}{2}m\omega^2 xy.$$

Consider λ to be a very small parameter and thus the second term in the potential could be treated as a perturbation.

(a) (5 points) What is the first order energy correction to the ground state? Obtain the answer by an argument based on parity.

(b) (5 points) Calculate the energy correction to the first excited energy level to the first order in perturbation theory.

Hint: For a one-dimensional harmonic oscillator, the lowering operator is given by

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right)$$

10. QM-4

A particle is moving in a spherically symmetric attractive potential $V(r) = -\lambda \frac{e^{-r}}{r}$, with $\lambda > 0$. The wave function for an energy eigenstate of definite angular momentum ℓ, $\ell = 0, 1, \cdots$, has the form $\Psi(r, \theta, \phi) = \frac{u_\ell(r)}{r} Y_\ell^m(\theta, \phi)$. The radial part of the time-independent Schrodinger equation reduces to an ordinary second order differential equation for $u_\ell(r)$,

$$\hat{H}_\ell u_\ell(r) \equiv \left[\frac{\hbar^2}{2\mu} \left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \right) - \frac{\lambda}{r} e^{-r} \right] u_\ell(r) = E \, u_\ell(r).$$

Consider the following trial wave-function,

$$u(r) = A r e^{-\kappa r},$$

with κ as a variational parameter. [A better trial function is $u(r) = Ar^2 e^{-\kappa r}$. To keep things simple, use the one given above.]

(a) (7 points) Find a variational estimate for the bound state energy, $E_1(\kappa)$, $E_1 < E_1(\kappa)$, and

$$E_1(\kappa) = \frac{\int_0^\infty dr u(r) \hat{H}_{\ell=1} u(r)}{\int_0^\infty dr u(r)^2} \equiv \langle -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} \rangle + \langle V_{eff}(r) \rangle$$

[Again, to simplify the problem, assume $\frac{\hbar^2}{2\mu} = 1$. to keep track of terms, evaluate $\langle -\frac{d^2}{dr^2} \rangle$ and $\langle V_{eff}(r) \rangle$ separately.]

(b) (3 points) Sketch $E_1(\kappa)$ as a function of κ. Show that, for λ sufficiently large, such a P-wave bound state indeed exists.

Useful Formula:

$$\int_0^\infty dr \, r^n e^{-cr} = c^{-(n+1)} n!$$