DEPARTMENT OF PHYSICS
BROWN UNIVERSITY
Written Qualifying Examination for the Ph.D. Degree
January 29, 2016

READ THESE INSTRUCTIONS CAREFULLY

1. The time allowed to complete the exam is 12:00–5:00 PM.
2. All work is to be done without the use of books or papers and without help from anyone. The use of calculators or other electronic devices is also not permitted.
3. Use a separate answer book for each question, or two books if necessary.
4. **DO NOT write your name in your booklets.** Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.
5. Write the problem number in the center of the outside front cover. **Write nothing else** on the inside or outside of the front and back covers. Note that there are separate graders for each question.
6. Answer one (and **only** one) problem from each of the five pairs of questions. The pairs are labeled as follows:

Classical Mechanics	CM1, CM2
Electricity and Magnetism	EM1, EM2
Statistical Mechanics	SM1, SM2
Quantum Mechanics	QM1, QM2
Quantum Mechanics	QM3, QM4

   Note that there are two pairs of Quantum Mechanics problems. You have to do **one** problem from **each** pair.

7. All problems have equal weight.

1. CM-1

A galaxy consists of many stars which are gravitationally bound. In a cluster of galaxies, over time stars from smaller galaxies are absorbed into larger ones. Consider such a smaller galaxy of mass m and radius r which passes in a straight line at constant speed v a larger galaxy of mass M and radius $r' > r$. *Estimate* the largest distance of closest approach R between the two galaxy centers such that the stars at the outer edge of the smaller galaxy are stripped from it.
You may assume that $R \gg r, r'$ and ignore all effects due to rotation of the galaxies.

HINT: Take the time the small galaxy feels the gravitational attraction of the larger one to be $\tau = R/v$.

2. CM-2

(a) A muon has a lifetime of $2\mu s$ in its rest frame. This muon is created 100km above the surface of the earth and move downwards at speed $0.99c$ where $c = 3 \times 10^8 ms^{-1}$. At what altitude does it typically decay? In the muon rest frame, how far does it travel relative to the earth?

(b) In an experiment at the Large Hadron Collider, a proton-proton collision event is recorded. Among the particles produced in this collision is an unusually high energy electron-positron pair (e^+e^-). The energies measured in the detector are $E(e^+) = 2.2\,TeV$ and $E(e^-) = 1.6\,TeV$. In the laboratory frame, the angle between the directions of the e^+ and the e^- is $40°$. Assuming the e^+e^- originated from the two-body decay of a new heavy particle: $Z' \to e^+e^-$, what is the mass of the Z'?

3. EM-1

If in a spherical coordinate system, the electric potential has the following form:

$$V(r) = \frac{qe^{-\alpha r}}{r}, \text{ , where } q \text{ and } \alpha \text{ are positive constants.}$$

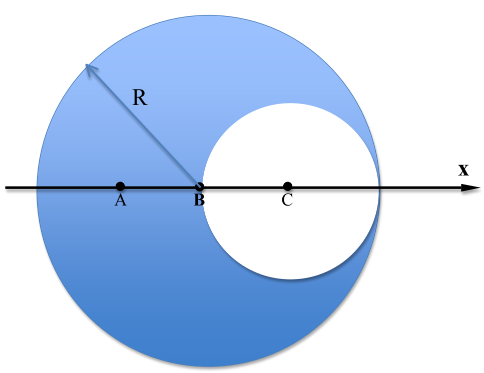
(a) Find the corresponding charge distribution.

(b) What is total charge in the system?

Note: In spherical coordinates, the Laplacian is:
$$\nabla^2 \equiv \frac{1}{r^2}\, r\big(r^2 \frac{\partial}{\partial_r}\big) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\big(\sin\phi \frac{\partial}{\partial\phi}\big).$$

4. EM-2

Starting with a uniformly charged sphere of radius R and volume charge density ρ, one carves out a spherical cavity of radius $R/2$ as shown in the drawing below. Calculate the electric field at positions A, B, and C, where B is at the center of the large sphere, A and C are of equal distance $R/2$ to the left and right of B, respectively.



5. SM-1

(a) (4 points) Consider a system of particles in 3-dimensional space that is described by a Hamiltonian $H(q, p)$ with q and p each running over $3N$ variables. Further assume that the potential is a function of the coordinates (q_i), so that

$$H(q, p) = \sum_{i=1}^{3N} \frac{p_i^2}{2m} + V(q_i).$$

Given the partition function $Z = \int d^3q d^3p e^{-H/kT}$, calculate the average value of $< p_x^2/2m >$, expressed in terms of temperature T.

(b) (4 points) Assume that the particles are confined to a finite region of space ($H \to \infty$ as any $q_i \to \infty$). Show that

$$\left\langle q_i \frac{\partial H}{\partial q_i} \right\rangle = kT.$$

(Hint: integrate by parts – the boundary term will vanish because of our condition). This is the "generalized equipartition theorem".

(c) (2 points) Given your results in parts (a) and (b), demonstrate that the classical "Virial theorem" holds for Hamiltonians of this form:

$$\left\langle p_i \frac{\partial H}{\partial p_i} \right\rangle = \left\langle q_i \frac{\partial H}{\partial q_i} \right\rangle.$$

6. SM-2

A colloidal particle in solution held in an optical trap obeys the following equation of motion,

$$0 = -\lambda \dot{x} - kx + f(t),$$

where λ is the viscous drag coefficent, k is the optical trap stiffness, and $f(t)$ is a random thermal force for which $\langle f(t) \rangle = 0$ and $\langle f(t) f(t') \rangle = \frac{2\lambda}{\beta}\delta(t - t')$. (Here $\langle ... \rangle$ denotes the ensemble average.) The solution to the equation of motion

(a) (2 points) Verify that the following expression is a solution to the equation of motion for $x(0) = 0$:

$$x(t) = \frac{1}{\lambda} e^{-\frac{k}{\lambda}t} \int_0^t e^{\frac{k}{\lambda}t'} f(t') dt'$$

(b) (4 points) Solve for the mean square displacement of the particle as a function of time, $\langle x(t)^2 \rangle$, after releasing the particle at the origin at time $t = 0$ (i.e. $x(0) = 0$).

(c) (4 points) Compute the power spectrum of the particle's fluctuations, $\frac{1}{T}\langle |\tilde{x}(\omega)|^2 \rangle$, for a particle in equilibrium whose position is measured over a time interval T. Sketch the power spectrum and indicate the characteristic frequency at which the spectrum changes from flat to $\tilde{x} \sim \omega^{-2}$.

($\tilde{x}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} e^{-i\omega t} x(t) dt$ is the Fourier transform of $x(t)$ for a measurement interval T.)

7. QM-1 Consider a non-relativistic spin-0 particle in a three-dimensional potential:

$$V(\vec{r}) = \frac{k}{2} r^2.$$

(a) Find the energy eigenvalues and determine the degeneracy of the first five of them. (7 points)

(b) Now suppose that there are six non-interacting particles placed in this potential. What is the ground energy of the system if these particles have spin (i) $\frac{1}{2}$; (ii) 1; (iii) $\frac{3}{2}$? (3 points)

8. QM-2 A "two-level" quantum system is one whose wave-function is given by a column vector, i.e., $|\phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$. The "Hamiltonian" for one such system is given by $\hat{H} = \lambda \hat{I} - \mu_1 \hat{\sigma}_x - \mu_2 \hat{\sigma}_y$, where $\hat{I}$ is the two-by-two identity matrix, $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Here λ, μ_1, and μ_2 are real and positive.

(a) What are the allowed energy eigenvalues? [Denote the eigenvalues by $E^{(i)}$, $i = 1, 2$.]

(b) What are the eigenvectors for this system? [Denote the corresponding eigen-vectors by $|E^{(i)}\rangle$, $i = 1, 2$.] To simplify life, let's introduce an angle α so that $\mu_1 = \mu \cos\alpha$ and $\mu_2 = \mu \sin\alpha$.

(c) At $t = 0$, the wave function is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find $|\phi(t)\rangle$ for $0 \le t$. Express you answer in terms of $E^{(1)}$, $E^{(2)}$, $|E^{(1)}\rangle$ and $|E^{(1)}\rangle$, etc.

(d) What is the probability of finding the system in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at $t > 0$?

9. QM-3 A particle of mass m moves in a one dimensional potential ranging from $x = 0$ to $x = a$,

$$V(x) = V_0 \cos\left(\frac{2\pi x}{a}\right), 0 \le x \le a.$$

Suppose the wave functions satisfy the anti-periodic condition over length a

$$\psi(x = a) = -\psi(x = 0).$$

(a) When $V_0 = 0$, the wavefunctions are simply plane waves $\psi(x) = e^{ikx}/\sqrt{a}$. Find the values of k that satisfy the anti-periodic condition in Eq. (2). What is the corresponding energy spectrum?

(b) Now consider a non-zero V_0 and V_0 is very small ($V_0 \ll \hbar^2/(ma^2)$). Calculate the energy shifts of the ground state to the first order in perturbation theory.

10. QM-4 Consider a simplified model of neutrino oscillations, where the neutrino oscillates between ν_e and ν_μ, represented by the mass eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$. The basis vectors are $H|\nu_1\rangle = E_1|\nu_1\rangle$ and $H|\nu_2\rangle = E_2|\nu_2\rangle$. The ν_e and ν_μ are combinations of the mass eigenstates such that $|\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle$ and $|\nu_\mu\rangle = -\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle$. At time $t = 0$, we start with a beam of pure ν_e of energy E. Calculate the probability that a ν_e has converted to a ν_μ after a time t.