

DEPARTMENT OF PHYSICS  
BROWN UNIVERSITY  
Written Qualifying Examination for the Ph.D. Degree  
January 23, 2015

READ THESE INSTRUCTIONS CAREFULLY

1. The time allowed to complete the exam is 12:00–5:00 PM.
2. All work is to be done without the use of books or papers and without help from anyone. The use of calculators or other electronic devices is also not permitted.
3. Use a separate answer book for each question, or two books if necessary.
4. **DO NOT write your name in your booklets.** Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.
5. Write the problem number in the center of the outside front cover. **Write nothing else** on the inside or outside of the front and back covers. Note that there are separate graders for each question.
6. Answer one (and **only** one) problem from each of the five pairs of questions. The pairs are labeled as follows:

Classical Mechanics	CM1, CM2
Electricity and Magnetism	EM1, EM2
Statistical Mechanics	SM1, SM2
Quantum Mechanics	QM1, QM2
Quantum Mechanics	QM3, QM4

Note that there are two pairs of Quantum Mechanics problems. You have to do **one** problem from **each** pair.

7. All problems have equal weight.

## 1. CM-1

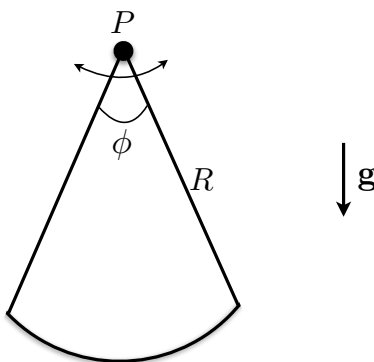
The Hamiltonian of a classical system with one degree of freedom has the form  $H = (ap^2 + bq^2)^2$ , where  $p$  and  $q$  are the canonical momentum and coordinate respectively.

- (a) (2 points) Write down Hamilton's equations.
- (b) (8 points) Solve Hamilton's equations with the initial conditions  $p(t = 0) = p_0$ ,  $q(t = 0) = q_0$ .

## 2. CM-2

A metal wedge with the shape of a slice of pizza is cut from a flat disc of uniform mass density (see figure below). The wedge rotates around the fixed point  $P$  while staying in the plane as shown below. The mass of the wedge is  $M$ , and the wedge hangs in a gravitational field  $g$ . The angle of the wedge is  $\phi$ , and the radius is  $R$ .

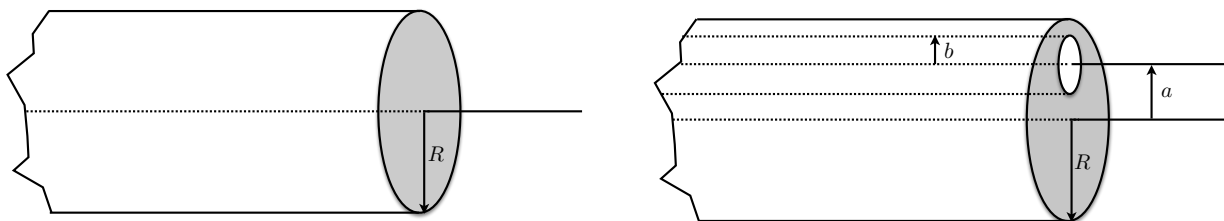
- (a) (6 points) Find the frequency of small oscillations.
- (b) (2 points) Calculate the length of a simple pendulum with the same frequency of oscillations.
- (c) (2 points) What is the length of the equivalent simple pendulum at the case of  $\phi \rightarrow 0$  and at the case of  $\phi = \pi$ .



### 3. EM-1

(i) (5 points) A very long uniform conducting cylinder of radius  $R$  carries a current  $I$  uniformly distributed over the cross section of the cylinder as shown in the left figure below. Find the magnetic field  $\mathbf{H}$  inside the cylinder at radius  $r < R$  from the long axis of the cylinder.

(ii) (5 points) A cylindrical hole of radius  $b$  is bored parallel to the long axis of the cylinder of part (i) with its center a distance  $a$  from the axis with  $a + b < R$  as shown in the right figure below. The current  $I$  is uniformly distributed across the conducting part of the cylinder. Find the magnetic field  $\mathbf{H}$  in this hole induced by the current flowing through the conducting part of the long cylinder with the hole. Draw a sketch of the field.

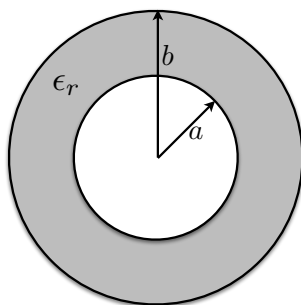


### 4. EM-2

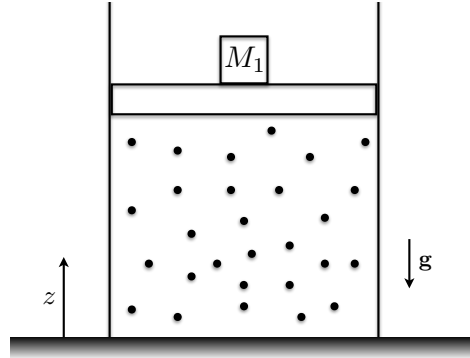
An uncharged conducting sphere of radius  $a$  is coated with a thick insulating shell (dielectric constant  $\epsilon_r$ ) out to radius  $b$  as shown below. The object is now placed in an external uniform electric field  $E_0$ . Find the electric field in the insulator.

Useful info:

$$V(r, \theta) = \sum_{\ell=0}^{\infty} \left( A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}} \right) P_{\ell}(\cos \theta)$$



## 5. SM-1



A cylinder contains an ideal monoatomic gas, supporting a moveable weightless piston on which there rests a mass of  $M_1$ . The region outside the cylinder is a perfect vacuum, and the height  $z$  of the piston is such that the gravitational force  $M_1g$  is exactly balanced by the pressure force. No heat can pass through the walls of the cylinder or the piston. The mass of the gas is small, so you may neglect the gravitational potential energy of the gas.

- (a) (5 points) If  $M_1$  is suddenly replaced by  $M_2 < M_1$ , the piston will come to a new equilibrium height  $z_2$ . Calculate  $z_2/z_1$  and the temperature ratio  $T_2/T_1$
- (b) (5 points) Now suppose  $M_1$  is diminished slowly to  $M_2$  by removing an infinitesimal mass element at each stage of the process. What are  $z_2/z_1$  and  $T_2/T_1$ ?

## 6. SM-2

- (a) Calculate the equation of state for a noninteracting two-dimensional Bose gas of particles with mass,  $m$  (no external potential). Leave your answer as an integral over the single-particle momentum  $p$ .
- (b) Calculate the average number of particles per unit area as a function of the chemical potential  $\mu$  and the temperature  $T$ .
- (c) Show that there is no Bose-Einstein condensation at nonzero temperatures in two dimensions.

## 7. QM-1

Consider a particle in a box of width  $2a$ . There is also a  $\delta$  function potential at the origin. The potential is given by:

$$V(x) = \begin{cases} \infty & ; |x| \geq a \\ \epsilon\delta(x) & ; |x| < a \end{cases}$$

(a) (4 points) First consider eigenfunctions which are even functions of  $x$ . Find these eigenfunctions in terms of the wave vector  $k$ ; do not worry about normalization.

(b) (2 points) Find the equation for  $k$  that determines the energy eigenvalues for the even functions; you do not need to solve this equation.

(c) (2 points) Solve the equation you found in part (b) when  $(\hbar^2/ma\epsilon) \ll 1$ ; you can assume that  $ka$  is of order one or smaller. Find the energy levels to leading order in  $(\hbar^2/ma\epsilon)$ .

(d) (2 points) Now consider the odd eigenfunctions. Solve for them (unnormalized) and the corresponding energy eigenvalues. Does the delta function play a role? Explain why or why not.

## 8. QM-2

Consider a spin 1/2 particle in a radially symmetric potential. Construct the wave functions in the  $|\ell, m\rangle$  basis that are eigenstates of  $J_z$ ,  $\mathbf{J}^2$  and  $\mathbf{L}^2$  simultaneously. Note that  $\mathbf{J} = \mathbf{L} + \mathbf{S}$  denotes the total angular momentum. You may find the following useful:  $L_{\pm}|\ell, m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}|\ell, m\pm 1\rangle$ .

### 9. QM-3

Consider a three-dimensional harmonic oscillator described by the Hamiltonian

$$H = \frac{|\mathbf{p}|^2}{2m} + \frac{1}{2}m\omega^2|\mathbf{x}|^2$$

- (a) (3 points) What are the energies and degeneracies of the three lowest levels?
- (b) (2 points) Classify these three lowest levels into states of fixed total angular momentum, to account for the degeneracy in part (a).
- (c) (5 points) Calculate the change in the ground state energy due to the perturbation

$$H' = \lambda(\mathbf{c} \cdot \mathbf{x})^3$$

where  $\mathbf{c}$  is a fixed vector, and  $\lambda$  is small. The correction should be calculated at second order in  $\lambda$ .

### 10. QM-4

Consider the potential:

$$U(x) = \begin{cases} \infty & ; |x| > a \\ \beta x^3 & ; |x| < a \end{cases}$$

If the energy  $E > \beta a^3$  then the WKB wave function assumes the form,

$$\psi(x) = \frac{C}{\sqrt{p(x)}} \sin\left(\frac{1}{\hbar} \int_{-a}^x p(y) dy\right).$$

- (a) (3 points) At  $E > \beta a^3$  derive the Bohr-Sommerfeld quantization rule. (Hint: The boundary conditions are different from the standard case of smooth potentials. Do not calculate the integral in your answer).
- (b) (2 points) Find the WKB energy levels at  $\beta = 0$ .
- (c) (5 points) For a small  $\beta$ , find the WKB levels with the accuracy of up to terms of order  $\beta^2$ , i.e., find the first 2 corrections in powers of  $\beta$  to the result of problem (b).