DEPARTMENT OF PHYSICS
BROWN UNIVERSITY
Written Qualifying Examination for the Ph.D. Degree
January 24, 2014

READ THESE INSTRUCTIONS CAREFULLY

1. The time allowed to complete the exam is 12:00–5:00 PM.
2. All work is to be done without the use of books or papers and without help from anyone. The use of calculators or other electronic devices is also not permitted.
3. Use a separate answer book for each question, or two books if necessary.
4. **DO NOT write your name in your booklets.** Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.
5. Write the problem number in the center of the outside front cover. **Write nothing else** on the inside or outside of the front and back covers. Note that there are separate graders for each question.
6. Answer one (and **only** one) problem from each of the five pairs of questions. The pairs are labeled as follows:

Classical Mechanics	CM1, CM2
Electricity and Magnetism	EM1, EM2
Statistical Mechanics	SM1, SM2
Quantum Mechanics	QM1, QM2
Quantum Mechanics	QM3, QM4

   Note that there are two pairs of Quantum Mechanics problems. You have to do **one** problem from **each** pair.
7. All problems have equal weight.

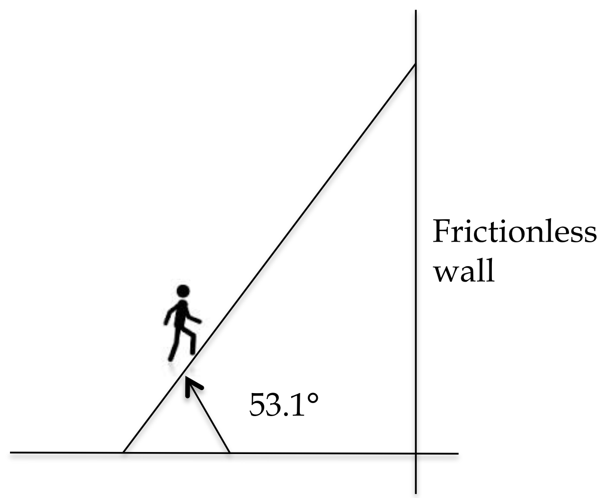
1. CM-1

Sir Lancelot is trying to rescue the Lady Elayne from Castle Von Doom by climbing a uniform ladder that is 5.0 m long and weighs 180 N. Lancelot, who weighs 800 N, stops a third of the way up to the ladder (See Figure below). The bottom of the ladder rests on a horizontal stone ledge and leans across the moat in equilibrium against a vertical wall that is frictionless because of a thick layer of moss. The ladder makes an angle of 53.1° with the horizontal, conveniently forming a 3-4-5 right triangle.

(a) Find the normal and friction forces on the ladder at its base.

(b) Find the minimum coefficient of static friction needed to prevent slipping at the base.

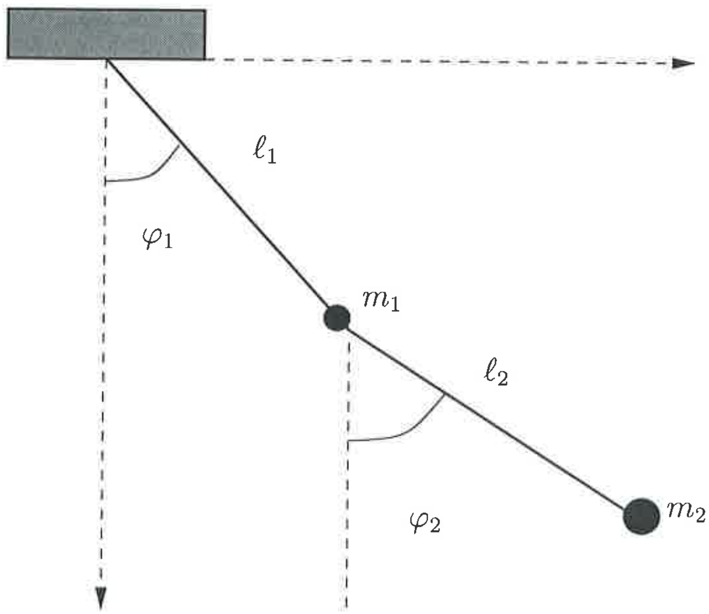
(c) Find the magnitude and direction of the contact force on the ladder at the base.



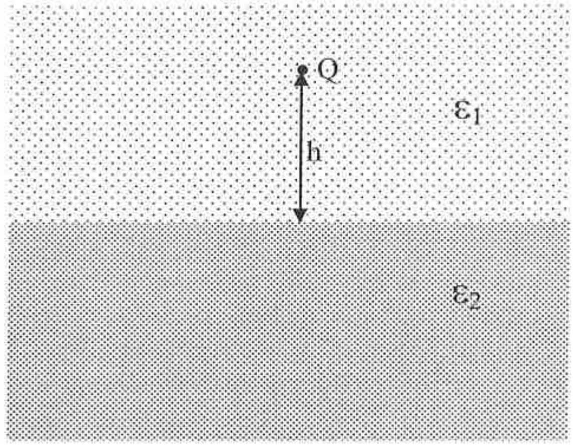
2. CM-2

(a) Find the Lagrangian for the coplanar double pendulum placed in a uniform gravitational field.

(b) Determine the frequencies of small oscillations. (To simplify, assume $\ell_1 = \ell_2 \equiv \ell$, and $m_1 = m_2 \equiv m$.)



3. EM-1



A point charge Q is embedded in a semi-infinite medium with dielectric constant ϵ_1, at a distance h from an interface with another semi-infinite medium, characterized by the dielectric constant ϵ_2.

(a) (8 points) Find the net force acting on the charge Q. Under what conditions is the charge repulsed from the interface between the two media?

(b) (2 points) Use the result of (a) to find the force between a point charge Q placed in vacuum at a distance h from an infinite metal place. Is the force repulsive or attractive?

4. EM-2

A conducting sphere of radius a with total charge Q is placed in an initially uniform field of magnitude E_0, pointing in the z-direction.

(a) Find the potential V everywhere outside the sphere.

(b) Find the charge density on the surface of the sphere.

5. SM-1

About 19 years ago, a group in Colorado succeeded in making a new sort of Bose-condensed gas. They cooled alkali atoms to a very low temperature in a magnetic trap. Suppose that a trap is made such that the potential experienced by an atom is

$$V(x, y, z) = \frac{1}{2}\alpha(x^2 + y^2 + z^2)$$

where α is a positive constant.

(a) What is the single particle density of states for particles in such a potential? Consider only energies E such that $E \gg \hbar\sqrt{\frac{\alpha}{m}}$, where m is the mass of an atom.

(b) If the total number of atoms in the trap is N, what is the Bose-Einstein condensation temperature?

Assume that the spin is zero. You may leave your final answer in terms of the integral,

$$I_n = \int_0^\infty \frac{x^n}{e^x - 1} dx.$$

6. SM-2

Consider a system of N classical harmonic oscillators on a lattice. Each oscillator has a natural frequency ν. Using the microcanonical ensemble find the entropy S as a function of N, ν and temperature T.

Useful information:

(1) The volume of a unit sphere in n-dimensional space is $\pi^{n/2}/\Gamma((n/2) + 1)$

(2) $\Gamma(x + 1) = x! \sim x^{x+1/2}e^{-x}\sqrt{2\pi}$

(3) $\log x! \sim x \log x - x$

7. QM-1

(a) A particle of mass m moves under the influence of a delta-function potential

$$V(x) = -\frac{g_0 \hbar^2}{2m}\delta(x).$$

With $g_0 > 0$, this potential is attractive and has one bound state. Denote the energy of the ground state by $E_0 = -\Delta_0$, where $\Delta_0 > 0$, and its wave function by $\phi_0(x)$. Find Δ_0 in terms of g_0, m, and $\hbar$.

(b) The potential is now modified by adding an extra repulsive piece,

$$V(x) = -\frac{g_0 \hbar^2}{2m}\delta(x) + \frac{g_1 \hbar^2}{2m}\delta(x-a),$$

When $g_1 > 0$ but *sufficiently small*, one expects that this potential can still support one bound state. The ground state wave function $\phi_0(x)$, now takes on the form $\phi_0(x) = Ae^{-kx} + Be^{kx}$ for the region $0 \le x \le a$. [The ratio A/B can be fixed by matching ϕ_0 and ϕ_0' at $x = 0$ and at $x = a$.] Obtain a transcendental equation which determines the new ground state energy $E_0' = -\Delta_0' < 0$. (i) Show that this equation leads to your earlier result for E_0 in the limit $g_1 \to 0$. (2) With g_1 fixed, show that a bound state should exist when a is very large.

8. QM-2

Consider three particles with spins $S_1 = S_2 = S_3 = 1$.

(a) (1 point) Find the dimension of the Hilbert space of this system.

(b) (2 points) What are possible values of the total angular momentum?

(c) (4 points) How many linearly independent states exist for each possible value of the total angular momentum?

(d) (3 points) Find the eigenvalues of the following Hamiltonian

$$\hat{H} = J(\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 + \hat{\vec{S}}_2 \cdot \hat{\vec{S}}_3 + \hat{\vec{S}}_3 \cdot \hat{\vec{S}}_1)$$

What is the degeneracy of each level?

9. QM-3

(a) (5 points) Prove the variational theorem that states that

$$\langle\psi|\hat{H}|\psi\rangle \geq E_0$$

where E_0 is the ground state energy, and $|\psi\rangle$ is any normalized state.

(b) (2 points) Consider the Hamiltonian for a particle moving in one dimension:

$$H = \frac{\hat{p}^2}{2m} + V_0(x/a)^6$$

where m is a mass, a is a length scale, and V_0 is an energy. Is the wavefunction

$$\psi(x) = C(x^2 - a^2)e^{-(x/d)^4},$$

where C is a normalization constant and d is an adjustable parameter with the dimension of length, a good choice for a variational approximation to the *ground state*? Why or why not?

(c) (3 points) If you answered in part (b) that it is a good choice, make a *rough* order-of-magnitude estimate of the optimal choice of d. If you instead answered that it isn't a good choice, propose a better variational wavefunction, including an estimate of the length scale.

10. QM-4

Suppose the electron had a small electric dipole moment that is proportional to the spin magnetic moment $\vec{\sigma}$, so that the Hamiltonian for the Hydrogen atom (neglecting relativistic effects) could be written as

$$H = \frac{p^2}{2m} - \frac{e^2}{r} - \gamma\vec{E}\cdot\vec{\sigma} = H_0 - \gamma\vec{E}\cdot\vec{\sigma}$$

where γ is the coupling strength, H_0 is the usual hydrogen atom Hamiltonian, and $\vec{E} = e\vec{r}/r^3$. (Recall that you can write $\vec{\sigma} = (\frac{2}{\hbar})\vec{S}$.)

(a) (5 points) Which of the following are constants of the motion for this Hamiltonian: $\vec{L}$, L^2, $\vec{S}$, S^2, $\vec{J}$, J^2?

(b) (2 points) Consider the case where γ is small. Use time-independent perturbation theory to show that the first order correction to the ground state energy vanishes. (3 points) Write down an expression for the second order perturbation to the ground state energy due to virtual transitions to the ^2P multiplet. (Calculate the spin matrix elements, but leave radial and angular momentum integrals in the final expression unevaluated).