DEPARTMENT OF PHYSICS
BROWN UNIVERSITY
Written Qualifying Examination for the Ph.D. Degree
January 27, 2012

READ THESE INSTRUCTIONS CAREFULLY

1. The time allowed to complete the exam is 12:00–5:00 PM.
2. All work is to be done without the use of books or papers and without help from anyone. The use of calculators or other electronic devices is also not permitted.
3. Use a separate answer book for each question, or two books if necessary.
4. **DO NOT write your name in your booklets.** Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.
5. Write the problem number in the center of the outside front cover. **Write nothing else** on the inside or outside of the front and back covers. Note that there are separate graders for each question.
6. Answer one (and **only** one) problem from each of the five pairs of questions. The pairs are labeled as follows:

Classical Mechanics	CM1, CM2
Electricity and Magnetism	EM1, EM2
Statistical Mechanics	SM1, SM2
Quantum Mechanics	QM1, QM2
Quantum Mechanics	QM3, QM4

Note that there are two pairs of Quantum Mechanics problems. You have to do **one** problem from **each** pair.

7. All problems have equal weight.

1. CM-1

In this problem you will show that a planet orbiting the Sun follows an **elliptical** path. You may assume that the mass of the Sun, M, is much greater than that of the planet, $M \gg m$. Follow the steps outlined below:

(a) (2 points) First explain why the planet travels in a plane by showing that the angular momentum $\vec{\ell}$ is constant.

(b) (3 points) Center the Sun at the origin of a spherical coordinate system and write down the Lagrangian L for a point mass m moving in a central potential $V(r) = -k/r$ where $k \equiv GMm$. Recall that $x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$, and $z = r\cos(\theta)$. Use the result of part (a) to set $\theta = \pi/2$ without loss of generality. Derive the equations of motion for $r(t)$ and $\phi(t)$ from your Lagrangian.

(c) (2 points) Transform the equation of motion for the radial coordinate $r(t)$ into a more suitable form. In particular, find the differential equation obeyed by $r(\phi)$. Hint: this can be accomplished by first **showing** that

$$\frac{d}{dt} = \frac{\ell_z}{mr^2}\frac{d}{d\phi}$$

and using the relation to replace the time derivatives with derivatives with respect to ϕ.

(d) (3 points) Conical sections are defined by

$$\frac{1}{r(\phi)} = C(1 + e\cos\phi)$$

where C is a constant and $0 \leq e < 1$ is the eccentricity of the elliptical orbit. Verify that this is a solution to the equation of motion that you found in part (c), and express the constant C in terms of k, m, and ℓ_z. Hint: the following relation can be used to simplify your work:

$$\frac{1}{r^2}\frac{dr}{d\phi} = -\frac{d}{d\phi}\left(\frac{1}{r}\right)$$

2. CM-2

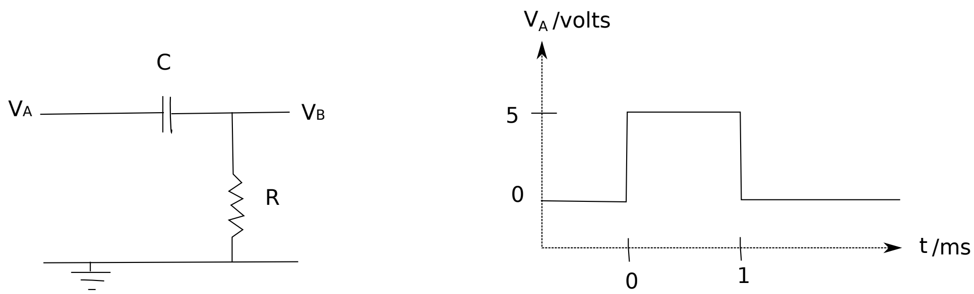
A pendulum consists of a mass m suspended from a fixed point by a rigid, massive rod of length ℓ and mass $M = 3m$. The density of the rod per unit length is constant, i.e., it is equal to M/ℓ. When the pendulum begins to swing, denote the angle made by the rod with respective to the vertical by θ. [Moment of inertia of a uniform thin rod of mass M and length ℓ about its CM is $M\ell^2/12$.]

(a) Find Lagrange's equation of motion for θ.

(b) Write down the approximate form of the equation of motion when θ is small. Specifically, write the equation to first order in θ. This is typically the case when ℓ is large and the horizontal displacement is small.

(c) Now assume that the pendulum is immersed in a viscous medium which contributes a term $3\sqrt{g/\ell}\frac{d\theta}{dt}$ to the equation for θ (i.e. add this term to the side of the equation of motion which has $\ddot{\theta}$ with no factor of m in front of it). If, at $t = 0$, $\theta = 0$ and $\dot{\theta} = 2\sqrt{g/\ell}$, find $\theta(t)$. Is the motion under-damped, critical-damped, or over-damped? [Solve the equation of motion using the ansatz: $\theta(t) = Ae^{\alpha t}$.]

3. EM-1



Consider the circuit in the figure composed of a resistor and a capacitor, with $R = 1k\Omega$ and $C = 1\mu F$.

(a) Find the differential equation that determines the output voltage V_B in terms of the input voltage V_A.

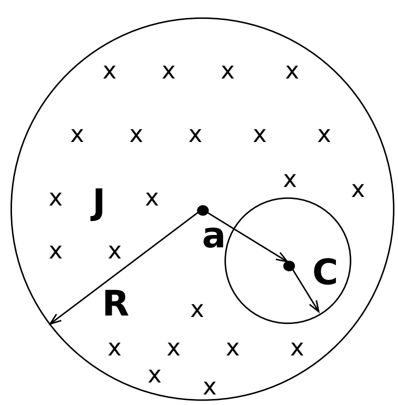
(b) Suppose the voltage pulse shown in the figure is applied at V_A, solve the differential equation to determine V_B for all times.

(c) Sketch V_B as a function of time

4. EM-2

A very long conducting cylinder of radius R has a hole of radius c parallel to the long axis cut out of it. The distance between the axes of the cylinder and the hole is a, where $a < R - c$.

Find the magnetic field **B** inside the hole when a uniform current density **J** parallel to the long axis flows in the remaining conducting part of the cylinder. Sketch the field **B** inside the hole.



5. SM-1

A linear polymer in water comprises N monomers and can exist in one of two states called "coil" and "helix". (The polymer performs a random walk in the coil state, whereas it folds into an ordered structure in the helix state.) The change in the Gibbs free energy of the total system when the molecule transitions from the coil state to the helix state is:

$$\Delta G_{c\to h} = (N-2)\left((E_w - E_m) - T\left(S_w - \frac{N}{N-2}S_p\right)\right), \tag{1}$$

where E_w is the binding energy between a monomer and the surrounding water in the coil state, E_m is the binding energy between two monomers in the helix state, S_w is the increase in entropy of the water per bound monomer in the helix state, S_p is the increase in the polymer's configurational entropy per monomer in the coil state, and T is the temperature. The factors of $N-2$ in the expression above account for end effects.

(a) (3 points) Determine the critical temperature, T_c, for the coil-helix transition. At T_c, the equilibrium probability of a molecule being in either state is $\frac{1}{2}$.

(b) (3 points) Assume that $2S_w - S_p > 0$ and $E_w - E_m > 0$, which is true for proteins that undergo a coil-helix transition. For what values of N is no transition expected to occur?

(c) (4 points) Determine the sharpness of the transition by computing $\frac{dP_h}{dT}|_{T=T_c}$, where P_h is the equilibrium probability of a molecule being in the helix state.

6. SM-2

Consider the three phases of water:

(a) (2.5 points) Sketch the phase diagram of water on the pressure (P) - temperature (T) plane.

(b) (2.5 points) Show that the latent heat of evaporation L_{vap} (the energy input by heating to cause one molecule to evaporate) is $L_{vap} = T(S_{vapor} - S_{liquid})$, where S_{vapor} and S_{liquid} are entropies per molecule in vapor and liquid phases respectively.

(c) (2.5 points) Show that along the liquid-vapor coexistence line, the chemical potentials of liquid and vapor are equal.

(d) (2.5 points) Prove the Clausius-Clapeyron relationship that links the latent heat of evaporation with the slope of the $P-T$ phase boundary as: $dP/dT = L_{vap}/T(v_{vap} - v_{liq})$, where v_{vap} and v_{liq} are specific volume (volume per molecule) of vapor and liquid respectively.

7.QM-1

Three identical quantum dots are placed in the corners of an equilateral triangle. An electron can hop between the quantum dots. The low-energy effective space of quantum states is three-dimensional. In the basis of the electron states, localized on each dot, the Hamiltonian is

$$\hat{H} = \begin{pmatrix} E_0 & t & t \\ t & E_0 & t \\ t & t & E_0 \end{pmatrix},$$

where E_0 is the on-site energy and t is the tunneling amplitude.

(a) (3 points) Use the fact that the above matrix is a Hamiltonian, and prove that t and E_0 are real.

(b) (7 points) How many energy levels does the system have? What is the degeneracy of each level? Find all energy levels.

8. QM-2

Consider an electron and positron with mass m and charges $-e$ and $+e$ respectively. Their relative motion is described by the Hamiltonian

$$H = \frac{\vec{P}^2}{2\mu} + V(r) + \frac{\lambda}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2$$

with $V = -\frac{e^2}{r}$ and λ constant. At time $t = 0$ the system is in the state

$$|\psi(0)\rangle = \phi_0(\vec{r})| + \frac{1}{2}, -\frac{1}{2}\rangle$$

where ϕ_0 is the ground state of $H_0 = \frac{p^2}{2\mu} + V(r)$, μ is the reduced mass, λ is a constant and the state is expressed in the basis $|S_{1z}, S_{2z}\rangle$.

(a) Calculate the magnetic moment of the system for all times $t \geq 0$.

(b) Find the ground state and first excited state of this system and their energies.

9. QM-3

(a) Write down the time dependent Schrodinger equation in momentum space for a free particle of mass m. Solve the equation and express your answer in terms of the initial state wave function $\Phi(p, t = 0)$.

(b) Find $\Phi(p, 0)$ for the Gaussian wave packet $\Psi(x, 0) = Ae^{-ax^2}e^{ilx}$ where l and a are real constants and determine $\Phi(p, t)$ in this case.

(c) Calculate $\langle p \rangle$ and $\langle p^2 \rangle$ for this wave packet as a function of time.

(d) Show that $\langle H \rangle = \langle p \rangle^2/2m + \langle H \rangle_0$ at any time, where $\langle H \rangle_0 = |A|^2 \int_{-\infty}^{+\infty} e^{-ax^2}\hat{p}^2 e^{-ax^2} dx$, and comment on the physical meaning of this result.

The following results should be useful:

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2 + \beta x} dx = e^{\beta^2/4\alpha} \left(\frac{\pi}{\alpha}\right)^{1/2}, \text{provided real part of } \alpha > 0$$

$$\int_{-\infty}^{+\infty} x^2 e^{-bx^2} dx = \frac{1}{2\alpha} \left(\frac{\pi}{b}\right)^{1/2}$$

10. QM-4

A particle mass m and charge q sits in a harmonic oscillator potential

$$V = \frac{k}{2}(x^2 + y^2 + z^2)$$

At time $t = -\infty$ the oscillator is in its ground state.

(a) (2 points) Express the harmonic oscillator Hamiltonian in terms of creation and annihilation operators and find its eigenvalues.

The system is then perturbed by a spatially uniform t-dependent electric field

$$\vec{E}(t) = Ae^{-(t/\tau)^2}\hat{z}$$

where A and τ are constants.
(b) (3 points) Compute the wave function at first order in perturbation theory by expressing the perturbation in terms of creation and annihilation operators.

(c) (5 points) Calculate in lowest-order perturbation theory the probability that the oscillator is in an excited state at $t = \infty$.