DEPARTMENT OF PHYSICS
BROWN UNIVERSITY
Written Qualifying Examination for the Ph.D. Degree
September 2022

READ THESE INSTRUCTIONS CAREFULLY

1. The time allowed to complete the exam is 10:00–3:00 PM.
2. All work is to be done without the use of books or papers and without help from anyone. The use of calculators or other electronic devices is also not permitted.
3. Use a separate answer book for each question, or two books if necessary.
4. **DO NOT write your name in your booklets.** Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.
5. Write the problem number in the center of the outside front cover. **Write nothing else** on the inside or outside of the front and back covers. Note that there are separate graders for each question.
6. Answer one (and **only** one) problem from each of the five pairs of questions. The pairs are labeled as follows:

Classical Mechanics	CM1, CM2
Electricity and Magnetism	EM1, EM2
Statistical Mechanics	SM1, SM2
Quantum Mechanics	QM1, QM2
Quantum Mechanics	QM3, QM4

   Note that there are two pairs of Quantum Mechanics problems. You have to do **one** problem from **each** pair.

7. All problems have equal weight.

1. CM-1

A roller coaster car of mass m moves along a frictionless track that lies in the x-y plane (where y is the vertical axis, and x is a horizontal axis. The height of the roller coaster along the track is given by $y = h(x)$. Note that the car stays attached to the track the whole time, so this is essentially a 1-d problem.

a) Using x as your coordinate, write down the Lagrangian (2 points) of the motion.

b) Use the Lagrangian to find the generalized momentum and the Hamiltonian for the car's motion in terms of the generalized momentum p and x. (3 points).

c) Use Hamilton's equations to find the equation for the acceleration of the car along the x direction. (3 points)

d) confirm your answer using Newton's second law. Write down an expression for the tangential force and tangential acceleration along the track, and recast the equation in terms of x. (2 points)

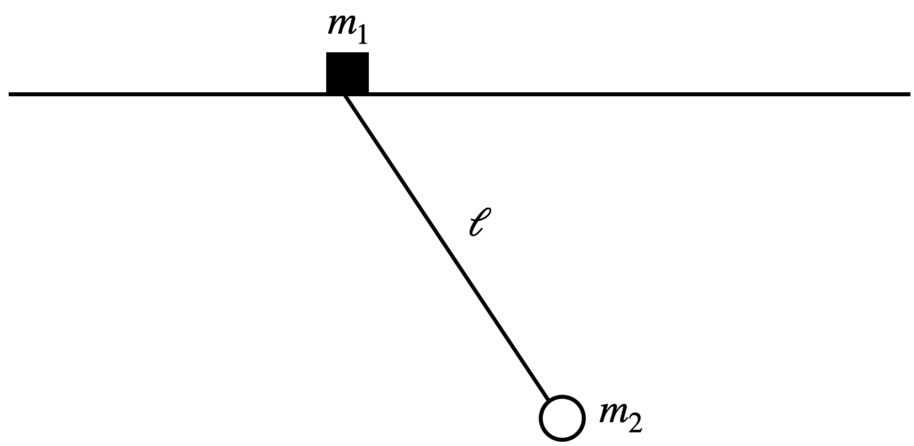
2. CM-2

A simple pendulum of mass m_2 and length ℓ is constrained to move in a single plane. The point of support is attached to a mass m_1 which can move on a horizontal line without friction in the same plane.

(a) (3 points) Find the Lagrangian of the system in terms of suitable generalized coordinates.

(b) (2 points) Derive the equations of motion.

(c) (5 points) Find the frequency of small oscillations of the pendulum. A complete answer will examine and comment upon the limits $m_2/m_1 \ll 1$ and $m_2/m_1 \gg 1$.



3. EM-1

A conductive metal sphere with radius R is connected to the earth ground. A charge q is placed at a distance $d > R$ away from the center of the sphere.

i Calculate the distribution of electric potential outside the sphere. Express the potential as a function of spatial coordinate (x, y, z) or (r, θ, ϕ).

ii if the sphere is not connected to the earth ground, but carries a known amount of charge Q, calculate the distribution of electric potential outside the sphere.

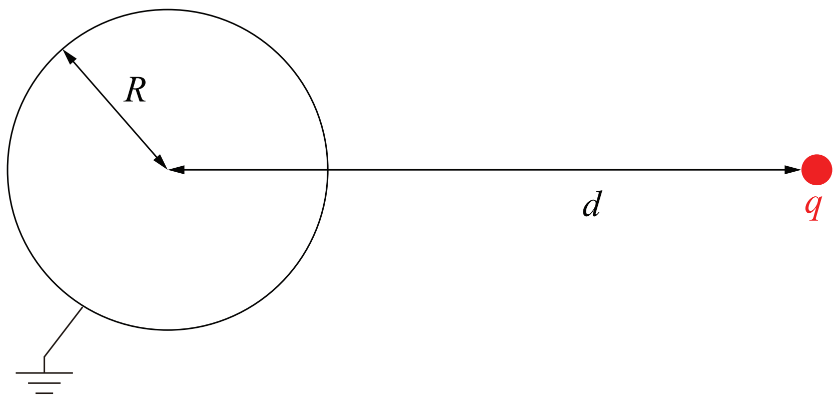


Figure 1: A conductive metal sphere with radius R is connected to the earth ground. A charge q is placed at a distance $d > R$ away from the center of the sphere.

4. EM-2

1. (1 point) A electric point charge A with charge q_A is at rest at the origin in system $\mathcal{S}$; a second point charge B with charge q_B flies by to the right at speed v on a trajectory parallel to the x axis, but at $y = d$. What is the electromagnetic force on charge B as it crosses the y axis?

2. (4 points) Now consider the same problem from system $\bar{\mathcal{S}}$, which moves to the right at speed v. What is the force on charge B when charge A crosses the $\bar{y}$ axis (i.e. when it is directly below charge B)? Use your answer to part 1 and transform the force to the new reference frame. Show your work.

3. (6 points) Again calculate the force on charge B when charge A crosses the $\bar{y}$ axis in the system $\bar{\mathcal{S}}$. This time, solve the problem by computing the fields in $\bar{\mathcal{S}}$ and use the Lorentz force law.

5. SM-1

Consider the Landau theory of phase transitions with the following Landau free energy $f(m)$, where m is the order parameter of the system:

$$f(m) = \frac{a}{2}m^2 + \frac{c}{3}m^3 + \frac{b}{4}m^4 \quad (1)$$

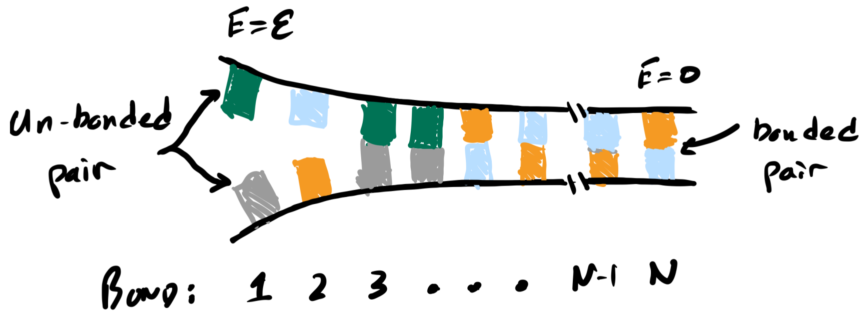
a. (6 points) Sketch f as a function of m for various values of a to show **qualitatively** how $\overline{m}$, the value or values of m that absolutely minimizes f, depends on a. **Assume that b and c are positive. Be sure to show enough sample values of a to indicate the possible phases and phase transitions. In each sketch, indicate with an arrow the point or points corresponding to $\overline{m}$.**

b. (3 points) Which of your sketches in part a corresponds to a first order phase transition? Find the values of $\overline{m}$ at the transition in terms of b and c.

c. (1 point) Find the value of a (in terms of b and c) where the first order phase transition occurs.

6. SM-2

(10 pts) DNA is formed from two helical molecular chains that are bound to each other through bonds between molecules called base pairs. The sequence of these pairs encodes information used to produce proteins. In order to access that information and perform a biological function, the two strands of a DNA molecule must be unwound.



Consider a simplified "zipper" model for the unwinding of a strand of DNA with N links of base pairs. Each base pair is either bonded with energy 0, or unbonded with energy ε, as shown in the figure above. Assume that the molecule can only unwind from the left side and that bond number k can only open if all bonds to the left $(1, 2, \ldots, k-1)$ are already open. Bonded pairs can only be in one configuration. However, the two pieces of any broken pair are free to rotate among g different positions.

a. (2 pts) Find an expression for the partition function of this system in terms of the quantity $x \equiv ge^{-\beta\varepsilon}$. Assume the last bonded base pair will never open. i.e. the chains cannot fall apart.

b. (2 pts) Find the average number of broken pairs as a function of x.

c. (2 pts) Consider the thermodynamic limit and find the average number of broken links for $x \ll 1$ and for $x \gg 1$.

d. (4 pts) Discuss what happens at $x = 1$? Write down a critical temperature corresponding to $x = 1$ and give a physical explanation for what happens here. Hint: consider the role of g.

7. QM-1

A measurement of the y component of the spin of an electron performed at time $t = 0$ has given the answer $+\hbar/2$.

(i) As soon as this experiment is completed a measurement of the z component of the spin of the electron is attempted. What is the probability that the result of this observation will be $\pm\hbar/2$?

(ii) Assume that the second measurement was performed and the result was $S_z = \hbar/2$. Assume also that a uniform, static magnetic field is present pointing in the direction of the positive y axis. What is the probability that a measurement of the x component of the spin performed at time $t \geq 0$ will give the answer $+\hbar/2$.

(iii) Under the conditions stated above calculate the expectation value of S_x at time $t \geq 0$.

8. QM-2

A particle of mass m propagates along the z-axis with the momentum $\hbar k$. It scatters off the potential $U[\delta(\vec{r}) - \delta(\vec{r} - \vec{e}_z a)]$, where $\vec{e}_z$ is a unit vector along the z axis. Find the differential scattering cross-section in the Born approximation. What happens at $a = 2\pi/k$ and $\theta = \pi/2$?

9. QM-3

Consider an asymmetric rigid rotor

$$H = \frac{1}{2I_0}(L_x^2 + L_y^2). \tag{2}$$

(i) What are its eigenfunctions and eigenvalues?

(ii) Enumerate all degenerate states.For low values of angular momenta $l = 1, 2$,is there additional degeneracy?

(iii) Consider the state

$$\Psi = x + y + z, \quad \text{with} \quad x^2 + y^2 + z^2 = 1. \tag{3}$$

Determine its time evolution $\Psi(t)$.

10. QM-4

Follow the steps outlined below to find the eigenenergies of a particle of mass m and charge q moving in two spatial dimensions in a uniform magnetic field described by the **symmetric** vector gauge potential:

$$\vec{A}(\vec{r}) = \frac{1}{2}\vec{B} \times \vec{r}\,.$$

(Do not use any other choice for the gauge.) You may assume that the particle moves in the x-y plane, and that the magnetic field points in the z-direction: $\vec{B} = B\hat{z}$.

a) (3 points) **Show** that the Hamiltonian may be written as the sum of two parts: $\hat{H} = \hat{H}_{sho} + \hat{H}_{\ell}$ where

$$\begin{aligned}\hat{H}_{sho} &= \frac{p_x^2 + p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2)\,, \\ \hat{H}_{\ell} &= \gamma B \hat{\ell}_z\,,\end{aligned}$$

and $\hat{\ell}_z = xp_y - yp_x$. **Solve for** γ **and** ω in terms of m, q, and B.

b) (2 points) The two pieces of the Hamiltonian commute: $[\hat{H}_{sho},\ \hat{H}_{\ell}] = 0$. **Explain why.** Evidently the eigenenergies of $\hat{H}$ are the sum of the eigenenergies of the two-dimensional simple harmonic oscillator (sho) and the eigenvalues of the z-component of the particle's angular momentum multiplied by a constant. The latter energy can be either positive or negative, so **can the eigenenergies of the full Hamiltonian $\hat{H}$ also be negative? Why or why not? Explain.**

c) (2 points) Are the eigenenergies degenerate? What is the ground state energy E_0?

d) (3 points) If $B = 10$ Tesla = 10^5 gauss and the charged particle is an electron with mass $m = 9.11\times10^{-31}kg$ and $q = -1.602\times10^{-19}C = -4.803\times10^{-10}esu$, what is E_0 in electron volts (eV)? Note that $1eV = 1.602\times10^{-19}J$ and $\hbar = 1.05 \times 10^{-34} J \cdot s$. (For simplicity you may ignore the Zeeman coupling of the electron spin to the magnetic field.) **Demonstrate that units combine in the dimensionally correct way to give an energy, and produce an actual numerical answer.**