DEPARTMENT OF PHYSICS
BROWN UNIVERSITY
Written Qualifying Examination for the Ph.D. Degree
September 2021

READ THESE INSTRUCTIONS CAREFULLY

1. The time allowed to complete the exam is 10:00–3:00 PM.
2. All work is to be done without the use of books or papers and without help from anyone. The use of calculators or other electronic devices is also not permitted.
3. Use a separate answer book for each question, or two books if necessary.
4. **DO NOT write your name in your booklets.** Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.
5. Write the problem number in the center of the outside front cover. **Write nothing else** on the inside or outside of the front and back covers. Note that there are separate graders for each question.
6. Answer one (and **only** one) problem from each of the five pairs of questions. The pairs are labeled as follows:

Classical Mechanics	CM1, CM2
Electricity and Magnetism	EM1, EM2
Statistical Mechanics	SM1, SM2
Quantum Mechanics	QM1, QM2
Quantum Mechanics	QM3, QM4

   Note that there are two pairs of Quantum Mechanics problems. You have to do **one** problem from **each** pair.
7. All problems have equal weight.

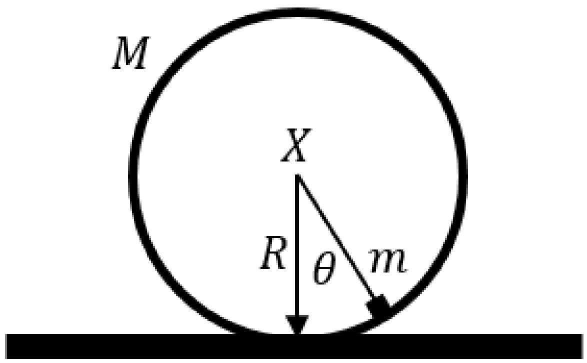


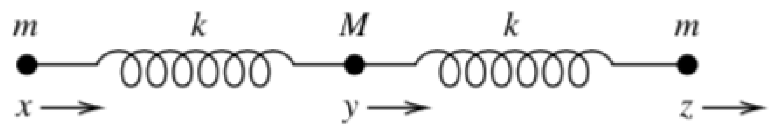
Figure 1: Caption

1. CM-1

[10 points] A hoop of mass M and radius R is free to roll without slipping on a horizontal surface. The mass of the hoop is distributed uniformly around its circumference. A small mass m is free to slide without friction on the inside surface of the hoop. Gravity acts downwards with a uniform acceleration g. What is the frequency of small oscillations of the mass around the bottom of the hoop?

2. CM-2

Three particles move in one dimension, as shown in the figure. The middle particle, at position y, has mass M and is connected by two identical springs, with spring constant k, to particles of mass m at positions x and y.

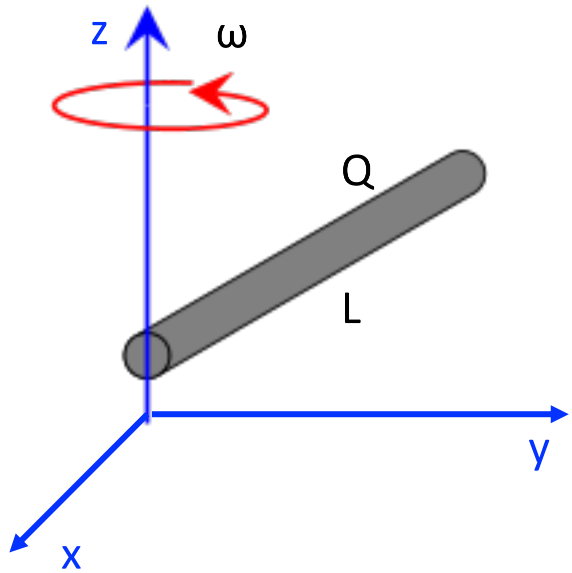


(a) [4 points] Write a Lagrangian for this system.

(b) [6 points] Find the normal modes (frequencies and displacements) of the system.

3. EM-1

An infinitesimally thin, uniformly charged rod of length L with a total electric charge of Q is rotating counterclockwise about the z-axis with a constant angular velocity $\omega \ll c/L$, as shown in the figure below. Here c is the speed of light. At time $t = 0$, the rod is parallel to the x axis and in the $x - y$ plane



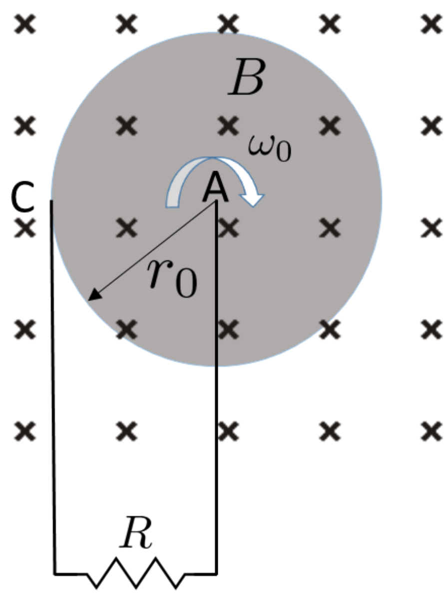
(a) [4 points] Find the electric dipole moment vector of the rod as a function of time.

(b)] 4 points] Find the magnetic dipole moment vector of the rod as a function of time.

(c) [2 points] Briefly comment, without calculating, on how the answers to (a) and (b) contribute to the power radiated by the rod.

4. EM-2

Consider a perfect conductor in the shape of a thin disk of radius r_0. The disk is in a uniform magnetic field of magnitude B indicated by the $\times$s, and the magnetic field is normal to the plane of the disk. Two wires with sliding contacts, make contact with the disk at the center (A) and at the edge of the disk (C) as shown. The wires are also connected as shown to a resistor with resistance R. You may assume the resistor is far away from the uniform magnetic field. The disk is rotated at a constant angular speed ω_0 such that the magnetic field remains normal to the plane of the disk.



(a) [4 points] In the limit that $R \to \infty$ no current will flow and electrons in the disk will rotate with constant angular speed. Calculate the Lorentz force on an electron at radius r.

(b) [4 points] In a steady state, this Lorentz force will be balanced by an electric field. Use this fact to calculate the EMF between points A and C.

(c) [2 points] Now assume R is large but finite. Calculate the current through the resistor, assuming it is entirely generated by the motion of the electrons.

5. SM-1

Consider a 1D chain of Ising spins with nearest neighbor interactions. The Hamiltonian is

$$H = -J \sum_{k=1}^{N} S_k S_{k+1} - h \sum_{k=1}^{N} S_k \tag{1}$$

where J is the exchange constant, and is assumed to be positive, $S = \pm 1$, h is the external magnetic field. We assume a periodic boundary condition, $S_{N+1} = S_1$.

(a) [4 points] Write down the partition function Z_N.

(b) [4 points] Derive the explicit expression for the partition function Z_N at large N. One approach is to write the partition function as a matrix product.

(c) [1 point] Calculate the Helmholtz free energy $F = -k_B T ln Z_N$.

(d) [1 point] Calculate the magnetic susceptibility (per spin) M/hN as $h \rightarrow 0$. Here M is the average total spin.

6. SM-2

(a) [4 points] Calculate the partition function at temperature T for a classical monoatomic ideal gas of N atoms of mass m in a uniform gravitational field g in a column of cross-sectional area A. The top of the column is at height h_2, and the bottom is at height h_1.

(b) [4 points] Find the pressure at the top and at the bottom of the column.

(c) [2 points] Give a physical interpretation of the the magnitude of the difference between the two pressures in part b.

7. QM-1

Consider a spin-1/2 particle of mass m and charge q in magnetic field $\vec{B}$.

(a) [4 points] Find the Hamiltonian describing the particle. The Pauli matrices are $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(b) [3 points] Consider the case of very weak magnetic field $\vec{B}$. Simplify the Hamiltonian from part (a) keeping the terms linear in magnetic field and discarding the all higher order terms. Use the symmetric gauge to do this calculation where $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$.

(c) [3 points] If we neglect the orbital motion of the particle the Hamiltonian takes form

$$H = -\frac{q\hbar}{2m}\vec{\sigma} \cdot \vec{B}$$

Calculate the groundstate and the groundstate energy for general $\vec{B}$.

8. QM-2

Consider a hydrogen atom at rest. We assume that the electron is in the radial ground state ($n = 0$) and the orbital angular momentum ground state ($\ell = 0$), and to begin with we consider only the Coulomb electric attraction between the electron and proton.

(a) [2 points] The electron wave function is given by $\psi = \mathcal{N}e^{-r/a_0}$ where a_0 is the Bohr radius and $\mathcal{N}$ is a normalization constant. Determine $\mathcal{N}$.

(b) [2 points] The total angular momentum (continuing to assume that the orbital angular momentum is zero) is

$$\mathbf{J} = \mathbf{S} + \mathbf{I}$$

where **S** and **I** are, respectively, the spin angular momenta of the electron and proton (each of which has spin-$\frac{1}{2}$). How many states does this system have and what are the $|S_z, I_z\rangle$ quantum numbers of these states?

(c) [5 points] Now consider, as a small perturbation to this system, the hyperfine interaction

$$H' = \frac{2\mu_0}{3}\frac{g_e\mu_B g_p\mu_N}{\hbar^2}\vec{\mathbf{S}}\cdot\vec{\mathbf{I}}\,\delta(\vec{\mathbf{r}})$$

Find the energy differences between the states you indicated in part (b). Be sure to clearly indicate which state(s) are now the ground state(s) (i.e., have lower energy) once this perturbation is taken into account.

(d) [1 point] When a hydrodgen atom transitions from the higher state(s) to the lower state(s), it emits a quantum of electromagnetic energy with the energy you calculated in part (c). Express the wavelength of this photon in units of centimeters to 1 significant figure, using the data below.

$$\text{Bohr radius } a_0 = 5.29 \times 10^{-11}\ \text{m}$$

$$\text{Bohr magneton } \mu_B = 9.27 \times 10^{-24}\ \text{A}\cdot\text{m}^2$$

$$\text{nuclear magneton } \mu_N = 5.05 \times 10^{-27}\ \text{A}\cdot\text{m}^2$$

$$\text{electron gyromagnetic ratio } g_e = 2.00$$

$$\text{proton gyromagnetic ratio } g_e = 5.59$$

$$hc = 1.99 \times 10^{-25}\ \text{kg m}^2\ \text{s}^{-2}$$

9. QM-3

Consider a one-dimensional harmonic oscillator with a Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2}m\omega^2\hat{x}^2 .$$

Let M be some fixed positive integer. For each integer k in the range $0, 1, \ldots M-1$, we define an angle $\theta_k = 2\pi\frac{k}{M}$ and a corresponding "phase state" $|\theta_k\rangle$ by the following equation:

$$|\theta_k\rangle = \frac{1}{\sqrt{M}}\sum_{n=0}^{M-1} e^{-in\theta_k}|n\rangle ,$$

where $|n\rangle$ is an energy eigenstate for the harmonic oscillator, i.e. $\hat{H}|n\rangle = \left(n + \frac{1}{2}\right)\hbar\omega|n\rangle$.

(a) [4 points] Compute the inner product $\langle\theta_k|\theta_m\rangle$ of two phase states when $k \neq m$ and $k = m$.

(b) [3 points] If our initial state $\left|\psi(0)\right\rangle = |\theta_k\rangle$, what is the new state $\left|\psi(t)\right\rangle$ at time t?

(c) [3 points] Compute the expectation value $\langle x\rangle = \left\langle\psi(t)\middle|\hat{x}\middle|\psi(t)\right\rangle$ (as a function of time) in the state $\left|\psi(t)\right\rangle$ from part (b).

Useful formula are provided below. Lowering and raising operators for harmonic oscillator:

$$\begin{aligned} \hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right), \\ \hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}_x\right), \end{aligned}$$

They act on the energy eigenstates of the harmonic oscillators:

$$\begin{aligned} \hat{a}^\dagger|n\rangle &= \sqrt{n+1}\,|n+1\rangle , \\ \hat{a}|n\rangle &= \sqrt{n}\,|n-1\rangle . \end{aligned}$$

10. QM-4

An ammonia molecule (NH_3) has a pyramidal structure, with 3 hydrogen atoms forming the base and 1 nitrogen at the apex. The nitrogen can be located either above or below the plane formed by the 3 hydrogen atoms. Call the state where N is above the plane $|1\rangle$ and the state where N is below the plane $|2\rangle$ and ignore any other internal degrees of freedom and translations/rotations of this molecule.

(a) [3 points] Write down the Hamiltonian, energy eigenvalues, and eigenstates for this molecule in free space when the potential separating states $|1\rangle$ and $|2\rangle$ is large, but not infinite. Call the eigenstates ψ_a and ψ_b, and the separation between energy eigenvalues $\hbar\omega_0$.

(b) [2 points] The distribution of valence electrons in the ammonia molecule results in a slight negative charge on the nitrogen atom and a slight positive charge on the hydrogen atoms. Thus, the NH_3 molecule has an electric dipole moment $\mu_{\mathbf{e}}$ directed away from nitrogen and towards the plane formed by the hydrogen atoms. Find the energy eignevalues for this system if a small static electric field, ϵ, is applied.

(c) [1 point] A small amplitude and time dependent electric field $\epsilon(t) = 2\epsilon_0 \cos(\omega t)$ is now applied. Write down a general form of the wavefunction for the ammonia molecule $\Psi(t)$ in this case.

(d) [4 points] Use perturbation theory to find the probability that a molecule that started out in state ψ_a at t=0 and subject to a time dependent field, will be found in state ψ_b at a later time t. Find this transition probability, $P_{a\to b}$ in the limit where ω is very close to ω_0.