

DEPARTMENT OF PHYSICS  
BROWN UNIVERSITY  
Written Qualifying Examination for the Ph.D. Degree  
September 2020

READ THESE INSTRUCTIONS CAREFULLY

1. The time allowed to complete the exam is 10:00–3:00 PM.
2. All work is to be done without the use of books or papers and without help from anyone. The use of calculators or other electronic devices is also not permitted.
3. Use a separate answer book for each question, or two books if necessary.
4. **DO NOT write your name in your booklets.** Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.
5. Write the problem number in the center of the outside front cover. **Write nothing else** on the inside or outside of the front and back covers. Note that there are separate graders for each question.
6. Answer one (and **only** one) problem from each of the five pairs of questions. The pairs are labeled as follows:

Classical Mechanics	CM1, CM2
Electricity and Magnetism	EM1, EM2
Statistical Mechanics	SM1, SM2
Quantum Mechanics	QM1, QM2
Quantum Mechanics	QM3, QM4

Note that there are two pairs of Quantum Mechanics problems. You have to do **one** problem from **each** pair.

7. All problems have equal weight.

## 1. CM-1

An intrepid explorer rides a hot-air balloon on a completely windless day. The balloon hovers fixed directly above a small target on the surface of the Earth. When she reaches a height  $h$ , she drops a small ball (of mass  $m$ ) straight down towards the target immediately below her (neglect air resistance in this problem!).

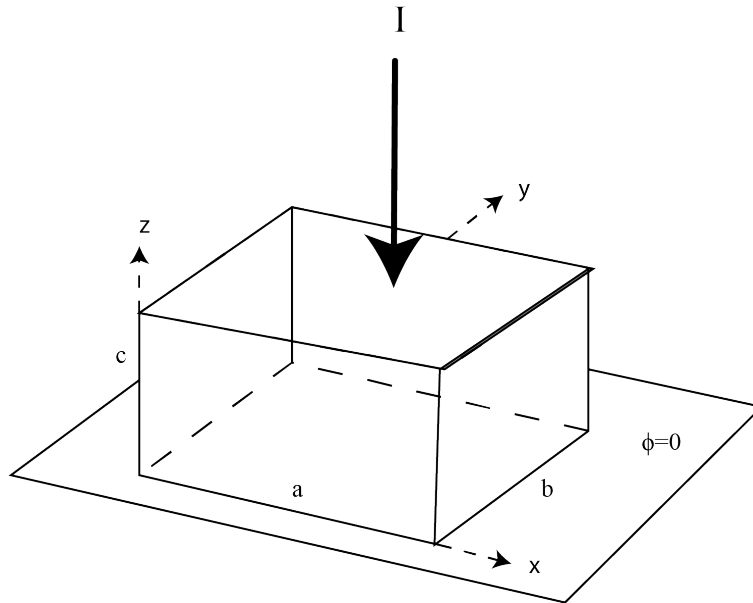
(a) [2 points] Where on Earth is she if the ball hits the center of the target exactly. Why does it miss if elsewhere?

(b) [3 points] Write down the Lagrangian corresponding to this system in the frame of the Earth, keeping in mind that the velocity as measured in an inertial (non-rotating) frame is related to the velocity in the rotating frame by  $\mathbf{v}_{\text{inertial}} = \mathbf{v}_{\text{rot}} + \boldsymbol{\Omega} \times \mathbf{x}$ , where  $\boldsymbol{\Omega}$  is the rotation rate. The gravitational field may be treated as a uniform field near the surface of the Earth.

(c) [5 points] If the rotation rate  $\boldsymbol{\Omega}$  is small, derive the equations of motion for the ball in the rotating frame to leading order in  $\boldsymbol{\Omega}$ .

## 2. CM-2

As a wheel rolls faster and faster on a flat, dirt-earth road, particles of mud are thrown from the rim to increased distances and heights. The forward speed of the wheel is  $v$  and the radius of the wheel is  $R$ . Assuming that the rolling speed is fast enough so that  $v^2 > Rg$ , where  $g$  is gravitational acceleration, derive an expression for the greatest height above the ground that the mud can go. Ignore the drag of the air on the mud particles. At what point on the rolling wheel does a mud particle leave in order to attain the highest point in the air?



### 3. EM-1

Consider steady current  $I$  flowing through a conducting block as shown in the figure. The conductor has conductivity  $\sigma$ . You may assume the relative permittivity  $\epsilon = 1$  and the relative permeability  $\mu = 1$ . The bottom face of the conductor is attached to a metal plate with vanishing resistance, where the potential vanishes. The other sides are surrounded by insulating material, except for the point of contact in the center of the face at  $z = c$ . Mathematically, current through this point of contact can be represented by a Dirac delta function in the current density.

(a) [1 point] Set up Laplace's equation in Cartesian coordinates for the electric potential  $\phi(x, y, z)$ .

(b) [2 points] As stated, the boundary conditions involve current and potential boundary values. Express these in terms of the potential using the relation between current density and electric field  $\vec{J} = \sigma \vec{E}$ .

(c) [4 points] Apply separation of variables to write an eigenfunction expansion for the solution with unknown constant coefficients.

(d) [3 points] Use the results of part (b) to solve for the unknown coefficients in part (c).

#### 4. EM-2

A circular disk of radius  $R$ , mass  $M$ , and moment of inertia  $I$ , carries  $n$  point, massless, charges (each with charge  $q$ ) attached at regular intervals around its rim.

The disk is immersed in a time-independent external magnetic field

$$\mathbf{B}(r, z) = k(-r\hat{\mathbf{r}} + 2z\hat{\mathbf{z}}),$$

where  $k$  is a constant.

At time  $t = 0$  the disk lies in the  $x - y$  plane, with its center at the origin, and rotates about the  $z$ -axis with angular velocity  $\omega_0$ .

(a) [2 points] Find the net force on the disk.

(b) [2 points] Find the net torque on the disk.

(c) [2 points] Find the it's angular velocity  $\omega(t)$ , as a function of time (ignore gravity).

(d) [2 points] Find the position of the center of the disk,  $z(t)$ .

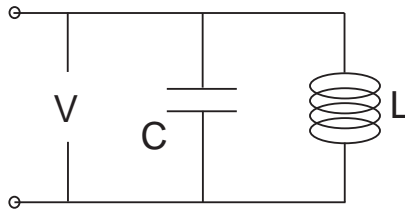
(e) [2 points] Describe the motion, and check that the total (kinetic) energy (translational plus rotational) is constant, confirming that the magnetic force does no work.

## 5. SM-1

An LC circuit is used as a thermometer by measuring the noise voltage across an inductor and capacitor in parallel. Treat the LC circuit as a *quantum* simple harmonic oscillator with energy

$$E = \frac{L}{2}\dot{q}^2 + \frac{1}{2C}q^2$$

where  $q$  is the charge. The voltage across the capacitor is  $V_C = q/C$  and the voltage across the inductor is  $V_L = L\dot{q}$ . Find the relation between the rms noise voltage ( $V_{rms}$ ) and the absolute temperature  $T$ . Find the limits of  $V_{rms}$  as  $T \rightarrow 0$  and as  $T \rightarrow \infty$ .



## 6. SM-2

Suppose you have a very large number,  $N$ , of distinguishable, non-interacting atoms each with just two, non-degenerate energy levels: 0 and  $\epsilon > 0$ . If the energy levels are non-degenerate and the average energy per atom is  $E/N$  in the limit that  $N \rightarrow \infty$  then

(a) [3 points] What is the maximum possible value of  $E/N$  if the system is not in equilibrium? What is  $E/N$  when the system is in thermal equilibrium with a heat bath, in the high temperature limit? Explain your reasoning.

(b) [3 points] Calculate the number of states available to the system when it has total energy  $E$  in terms of  $E$ ,  $N$ , and  $\epsilon$ .

(c) [4 points] Calculate the entropy per atom  $S/N$  of the system as a function of  $E/N$ . Provide a check of whether it agrees with your answer in a). (You may find  $\ln N! \approx N \ln N$  helpful)

## 7. QM-1

Two identical bosons are placed in an infinite square well

$$V(x) = \begin{cases} 0 & x \in [0, a] \\ \infty & \text{otherwise} \end{cases}$$

They interact weakly w/one another via the potential

$$V(x_1, x_2) = -aV_0\delta(x_1 - x_2)$$

**(a)** [3 pts] Ignoring the interaction, find the ground state and first excited state wavefunctions and energies.

**(b)** [7 pts] Use first-order perturbation theory to calculate the effect of the particle-particle interaction on the ground and first excited state energies.

### Useful Integrals

$$\int_0^a \sin^2 \frac{\pi x}{a} dx = \frac{a}{2}$$

$$\int_0^a \sin^4 \frac{\pi x}{a} dx = \frac{3a}{8}$$

$$\int_0^a \sin^6 \frac{\pi x}{a} dx = \frac{5a}{16}$$



## 8. QM-2

Consider a particle of mass  $m$  on a ring of circumference  $L$ .

(a) [2 points] Write down the Hamiltonian, and the Schrodinger equation with appropriate boundary conditions.

(b) [3 points] Determine eigenstates of the Hamiltonian and the energy levels.

(c) [2 points] Assume that the particle (on the same ring) has an electric charge  $q$  in the units of the electron charge. The ring is pierced by the magnetic flux  $\Phi$ . Write down the Hamiltonian and the Schrodinger equation.

(d) [3 points] Find eigenstates and energy levels of this system.

### 9. QM-3

A particle of mass  $m$  is confined to move along  $x$ -axis in the presence of a potential:  $V(x) = -V_0\delta(x)$ ,  $V_0 > 0$ .

(a) [5 points] Calculate the energy of bound states, and write down their wave functions.

(b) [5 points] Consider a right-moving plane wave with energy  $E > 0$  approaching the potential from  $x \rightarrow -\infty$ . Calculate the reflection and transmission coefficients.

### 10. QM-4

In this problem you will prove that the ground state of the one-dimensional Hamiltonian,

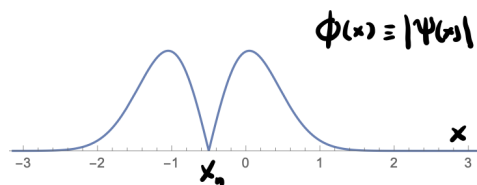
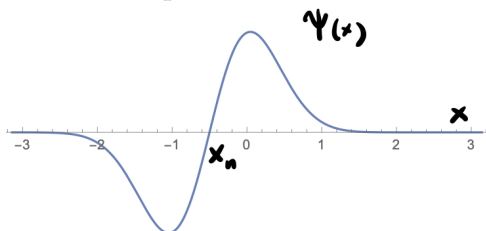
$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{X}),$$

where  $V(x)$  is a smooth potential with no singularities, **has no nodes**. Nodes are where wavefunctions change sign from positive to negative, or vice-versa. Note that since  $\hat{H}$  is Hermitian,  $V(x)$  must be real, so you can work in a basis in which all the eigenstates of  $\hat{H}$  are purely real.

To prove the theorem, assume the contrary, and show that this leads to a contradiction. Specifically, assume for simplicity that the purported ground state  $\psi(x)$  has a single node at location  $x = x_n$  where the wavefunction  $\psi(x)$  changes sign; thus  $\psi(x_n) = 0$ . (The generalization to more than one node is straightforward.) Further assume that  $\psi$  has already been normalized.

(a) [4 points] Compare the expectation value of the energy,  $\langle \psi | \hat{H} | \psi \rangle$ , to the expected energy of the state  $\phi(x) \equiv |\psi(x)|$  which of course has no nodes. Hints: Since  $\psi(x)$  is normalized, then so is  $\phi(x)$  because the absolute value doesn't change the normalization and thus you need only evaluate  $\langle \phi | \hat{H} | \phi \rangle$ . Do the two states  $\psi$  and  $\phi$  have the same, or different, energy expectation values?

(b) [3 points] Now show that  $|\phi\rangle$  cannot be an eigenstate of the Hamiltonian, because  $\phi(x)$  has a cusp at  $x = x_n$ . Cusps are where the first spatial derivative of the wavefunction changes discontinuously (see figure). Hint: show that cusps in a ground-state wavefunction imply that  $|\phi\rangle$  can only be an energy eigenstate for a potential  $V(x)$  that is not smooth and has a divergence.



(c) [3 points] Invoke the variational theorem that says  $\langle \psi | \hat{H} | \psi \rangle \geq E_0$ , where  $E_0$  is the ground state energy and equality is only realized when  $|\psi\rangle$  is a ground state, to explain why the initial assumption that the ground state has nodes leads to a contradiction.