DEPARTMENT OF PHYSICS
BROWN UNIVERSITY
Written Qualifying Examination for the Ph.D. Degree
September, 2019

READ THESE INSTRUCTIONS CAREFULLY

1. The time allowed to complete the exam is 10:00–3:00 PM.
2. All work is to be done without the use of books or papers and without help from anyone. The use of calculators or other electronic devices is also not permitted.
3. Use a separate answer book for each question, or two books if necessary.
4. **DO NOT write your name in your booklets.** Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.
5. Write the problem number in the center of the outside front cover. **Write nothing else** on the inside or outside of the front and back covers. Note that there are separate graders for each question.
6. Answer one (and **only** one) problem from each of the five pairs of questions. The pairs are labeled as follows:

Classical Mechanics	CM1, CM2
Electricity and Magnetism	EM1, EM2
Statistical Mechanics	SM1, SM2
Quantum Mechanics	QM1, QM2
Quantum Mechanics	QM3, QM4

   Note that there are two pairs of Quantum Mechanics problems. You have to do **one** problem from **each** pair.

7. All problems have equal weight.

1. CM-1

Consider a diatomic molecule with reduced mass m that is free to rotate around an axis perpendicular to the vector between the molecules. Let r_0 be the equilibrium separation of the atoms when at rest, and let ω_0 be the frequency of radial oscillations around this equilibrium.

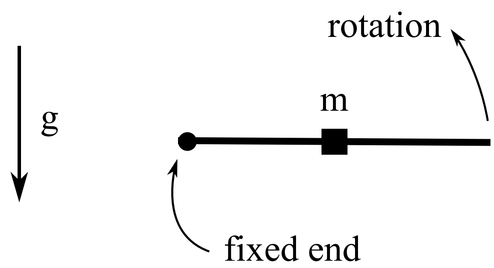
(a) (3 points) In terms of the quantities given, as well as $\dot{r}$ and $\dot{\theta}$, write an expression for the energy of this system.

(b) (3 points) Find the first order correction to the equilbrium separation when the angular momentum L is small but nonzero.

(c) (4 points) Similarly, find the first order correction to the frequency of radial oscillations.

2. CM-2

A thin rod of length L has one endpoint fixed and rotates with constant angular velocity ω in a vertical plane in the presence of a uniform gravitational field g. At $t = 0$ the rod is horizontal and moving upward. A particle of mass m slides without friction along the rod. The figure shows a snapshot of the system at $t = 0$:



(a) (3 points) Find the Lagrangian and equation of motion for the mass m.

(b) (4 points) Solve the differential equation to find the motion of the mass, assuming that at $t = 0$ the mass is at rest (relative to the rod) at a distance $d = L/2$ from the fixed end of the rod.

(c) (3 points) Show that there is a critical frequency ω_c such that the mass can fly off the rotating end of the rod only if $\omega > \omega_c$.

Note: for part (c) you can assume that if the mass hits the fixed end of the rod it becomes stuck there. In this way you don't have to worry about the possibility that the mass could hit the fixed end and then some time later slide off the rod.

3. EM-1

A steady current I flows through a long straight wire with resistance R. Where necessary you can use L to denote the length of the wire, but you can neglect edge effects.

(a) (3 points) Compute the Poynting vector $\vec{\mathbf{S}}$ outside the wire. Be sure to clearly indicate its direction.

(b) (3 points) Calculate the total energy flux at the surface of the wire. Be sure to clearly indicate the sign of the energy flux (specifically, is energy flowing into or out of the wire).

(c) (4 points) Use Poynting's theorem

$$-\frac{\partial U}{\partial t} = \nabla \cdot \vec{\mathbf{S}} + \vec{\mathbf{J}} \cdot \vec{\mathbf{E}}$$

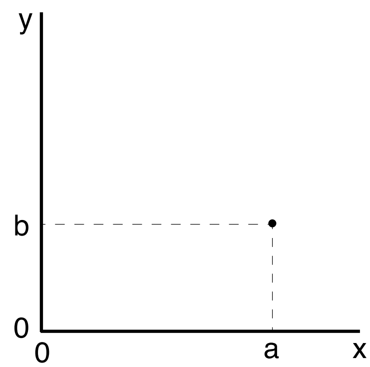
(where U is the energy density of the electromagnetic field) to show that the rate at which the fields do work on the charges in the wire (i.e., the power dissipated by the wire) is $P = I^2 R$.

4. EM-2

Two infinite planes of conductor are aligned perpendicular to each other as shown in the figure. The two planes are located at $x = 0$ and $y = 0$, respectively. Initially, the conducting planes are grounded.

(a) (4 points) if a charged particle with charge q is placed at $x = a$ and $y = b$, what will be the force on this charged particle ?

(b) (3 points) if the charged particle is moved slowly from $x = a$ and $y = b$ to infinitely far away, what is the work done by the external force



(c) (3 points) Now imagine that the planes are not infinite but very large, with dimension $L \gg a, b$, and are connected to ground by an ammeter. Sketch a plot of the current measured by the ammeter as a negative charge q is moved to infinity as in part (b).

5. SM-1

Van der Waals equation of state

$$\left(p + a\frac{N^2}{V^2}\right)(V - Nb) = NkT$$

is a reasonable approximation to describe the liquid/gas phases of many substances. Here a and b are constants, and the other symbols have their standard meaning.

(a) (2 points) Sketch the interaction potential between two molecules. Referring to this, justify the two new terms that appear in the van der Waals equation of state, versus that of the ideal gas.

(b) (2 points) Sketch isotherms of the equation of state and label the liquid and gas regions, and the critical point. Explain what the critical point is.

(c) (2 points) Outline the arguments that allow one to solve for the position of the critical point at $kT_c = 8a/27b$, $p_c = a/27b^2$, $V_c = 3Nb$.

(d) (2 points) For a general gas, the equation of state may be expanded in a virial expansion

$$\frac{p}{kT} = \frac{N}{V} + B_2\left(\frac{N}{V}\right)^2 + B_3\left(\frac{N}{V}\right)^3 + \cdots$$

where the B_n are virial coefficients. Under what circumstances may the higher order terms be ignored?

(e) (2 points) Find B_2 for the van der Waals gas and sketch its variation with temperature. The Boyle temperature T_B is the temperature at which this goes to zero. Solve for T_B in terms of T_c.

6. SM-2

(a) (5 points) Calculate the equation of state for a two-dimensional Bose gas. Leave your answer as an integral over the single-particle momentum p.

(b) (3 points) Calculate the average number of particles per unit area as a function of the chemical potential and the temperature T.

(c) (2 points) Show that there is no Bose-Einstein condensation at nonzero temperatures in two dimensions.

7. QM-1

Identical non-interacting particles of mass m move on a ring of radius R. The Hamiltonian is

$$\hat{H} = -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2}$$

where $0 \leq \theta \leq 2\pi$ and the wave function satisfies the periodic boundary conditions $\psi(0) = \psi(2\pi)$, $\psi'(0) = \psi'(2\pi)$.

(a) (3 points) Find the energy levels and their degeneracies for a single spinless particle.

(b) (1 point) Find the ground state energy and degeneracy for three identical spinless bosons with the above Hamiltonians.

(c) (2 points) Answer question **b)** for the first excited state.

(d) (2 points) Find the ground state energy and degeneracy for three identical spin-1/2 fermions with the above Hamiltonian.

(e) (2 points) Answer question **d)** for the first excited state.

8. QM-2

Consider the effect of an electric field E on a charged particle of charge q and mass m bound to a harmonic trap. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + qEx$$

and x and p satisfy the canonical commutation relation:

$$[x, p] = i\hbar$$

(a) (2 points) For zero electric field $E = 0$, the Hamiltonian can be written as:

$$H_0 = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2}),$$

derive or write down the explicit forms for operators $\hat{a}$ and $\hat{a}^\dagger$.

(b) (3 points) Find ground state and first excited state wave functions and energy eigenvalues for H_0.

(c) (5 points) For nonzero electric field E, find the change in the ground state and the first excited state energy eigenvalues.

9. QM-3

Consider the following operators

$$A_1 = -2\left(z^{\frac{1}{2}}\frac{\partial}{\partial z} + z^{\frac{3}{2}}\frac{\partial}{\partial z}\right) \tag{1}$$

$$A_2 = 2\left(z^{\frac{1}{2}}\frac{\partial}{\partial z} - z^{\frac{3}{2}}\frac{\partial}{\partial z}\right) \tag{2}$$

$$A_3 = 4z\frac{\partial}{\partial z} \tag{3}$$

(a) (2 points each, 6 points total) Calculate all the commutation relation between these operators, $[A_1, A_2], [A_1, A_3], [A_2, A_3]$.

(b) (4 points) Imagine that z is a coordinate, in the complex plane, of a particle on a ring of radius 1, $z = e^{i\theta}$. Find the eigenvalues and eigenstates of A_3.

10. QM-4

Consider a hydrogen molecule comprised of two electrons (of mass m) at coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ and two protons at fixed positions $\mathbf{R}_a$ and $\mathbf{R}_b$. The Hamiltonian can be written as

$$\mathcal{H} = \mathcal{H}_a + \mathcal{H}_b + \mathcal{H}' ,$$

where

$$\mathcal{H}_a = \frac{p_1^2}{2m} - \frac{e^2}{|\mathbf{r}_1 - \mathbf{R}_a|} \quad \text{and} \quad \mathcal{H}_b = \frac{p_2^2}{2m} - \frac{e^2}{|\mathbf{r}_2 - \mathbf{R}_b|}$$

describe two individual hydrogen atoms, and the terms for repulsion or attraction of the constituents of one atom with those of another

$$\mathcal{H}' = \frac{e^2}{|\mathbf{R}_a - \mathbf{R}_b|} + \frac{e^2}{|\mathbf{r}_1 - \mathbf{r}_2|} - \frac{e^2}{|\mathbf{R}_a - \mathbf{r}_2|} - \frac{e^2}{|\mathbf{R}_b - \mathbf{r}_1|}$$

are treated as a perturbation.

(a) (6 points) If we were to ignore the perturbation $\mathcal{H}'$, the ground states of the molecule would have energy $2E_0$, where E_0 is the ground state energy of each individual atom. Find the corrections to the ground state energies to first order in perturbation theory. Express your answer in terms of the Coulomb integral K, exchange integral J, and overlap integral S defined by

$$K = \int \phi_a^*(\mathbf{r}_1)\phi_b^*(\mathbf{r}_2)\mathcal{H}'\phi_a(\mathbf{r}_1)\phi_b(\mathbf{r}_2)d\mathbf{r}_1 d\mathbf{r}_2,$$
$$J = \int \phi_a^*(\mathbf{r}_1)\phi_b^*(\mathbf{r}_2)\mathcal{H}'\phi_b(\mathbf{r}_1)\phi_a(\mathbf{r}_2)d\mathbf{r}_1 d\mathbf{r}_2,$$
$$S = \int \phi_a^*(\mathbf{r}_1)\phi_b^*(\mathbf{r}_2)\phi_b(\mathbf{r}_1)\phi_a(\mathbf{r}_2)d\mathbf{r}_1 d\mathbf{r}_2,$$

where $\phi_{a,b}$ are the normalized ground state wavefunctions of $\mathcal{H}_a, \mathcal{H}_b$.

Hint: Use the following symmetrized, unnormalized, two particle wavefunctions

$$\psi_\pm = \frac{1}{\sqrt{2}}\left[\phi_a(\mathbf{r}_1)\phi_b(\mathbf{r}_2) \pm \phi_b(\mathbf{r}_1)\phi_a(\mathbf{r}_2)\right]$$

(b) (4 points) Argue that the energy difference between the triplet and singlet electronic states in the molecule is given by

$$E_s - E_t = \frac{2J - 2KS}{1 - S^2}.$$