

DEPARTMENT OF PHYSICS
BROWN UNIVERSITY
Written Qualifying Examination for the Ph.D. Degree
September, 2017

READ THESE INSTRUCTIONS CAREFULLY

1. The time allowed to complete the exam is 10:00–3:00 PM.
2. All work is to be done without the use of books or papers and without help from anyone. The use of calculators or other electronic devices is also not permitted.
3. Use a separate answer book for each question, or two books if necessary.
4. **DO NOT write your name in your booklets.** Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.
5. Write the problem number in the center of the outside front cover. **Write nothing else** on the inside or outside of the front and back covers. Note that there are separate graders for each question.
6. Answer one (and **only** one) problem from each of the five pairs of questions. The pairs are labeled as follows:

Classical Mechanics	CM1, CM2
Electricity and Magnetism	EM1, EM2
Statistical Mechanics	SM1, SM2
Quantum Mechanics	QM1, QM2
Quantum Mechanics	QM3, QM4

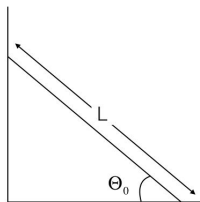
Note that there are two pairs of Quantum Mechanics problems. You have to do **one** problem from **each** pair.

7. All problems have equal weight.

1. CM-1

A uniform ladder of mass M and Length L is placed with one end against a frictionless floor and the other end on a frictionless wall. Initially the ladder makes an angle Θ_0 with the floor (see figure below).

When the ladder is released, it slides under the influence of gravity. At some angle the ladder detaches from the wall.



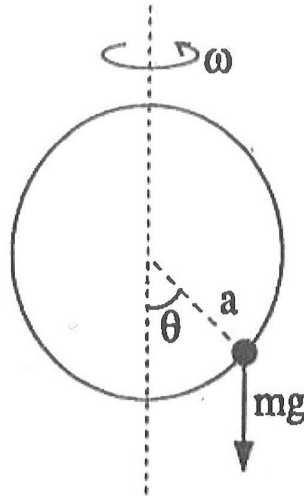
- (a) (5 points) Write down the Lagrangian as a function of the angle, Θ (the angle of the ladder with respect to the floor) before it detaches from the wall.

Note: Recall that the moment of inertia of a uniform rod of Length L and mass M rotating about an axis through its center of mass is $I = \frac{1}{12}ML^2$

- (b) (5 points) Find the angle Θ_1 at which the ladder detaches from the wall.

2. CM-2

A bead of mass m slides without friction on a circular loop of radius a . The loop lies in a vertical plane and rotates about a vertical diameter with constant angular velocity ω .

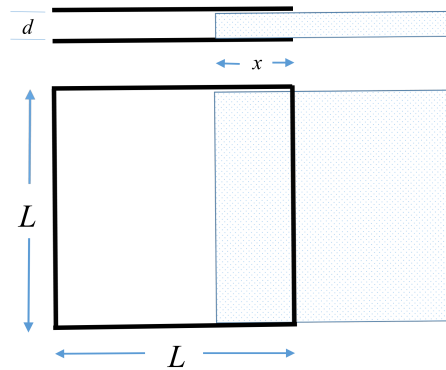


- (a) (6 points) For angular velocity ω greater than some critical angular velocity ω_c , the bead can undergo small oscillations about some stable equilibrium point $\theta_0 > 0$. Find ω_c and $\theta_0(\omega)$.
- (b) (4 points) Obtain the equations of motion for the small oscillations about θ_0 as a function of ω and find the period of the oscillations.

3. EM-1

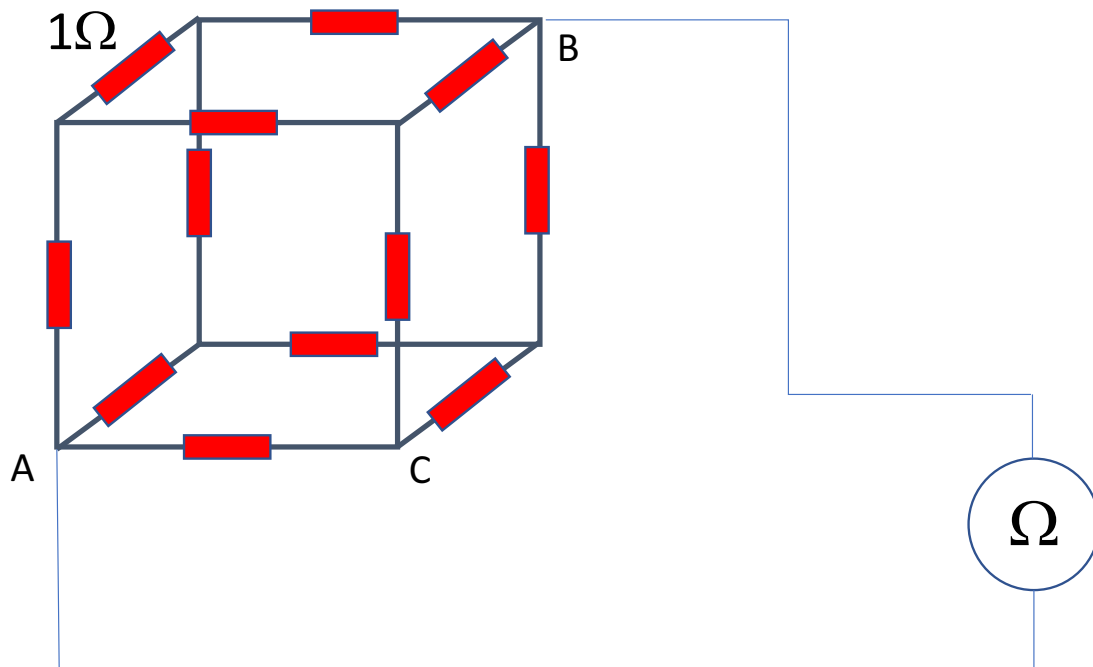
1. (10 points) Two square metal plates with length and width L are separated by a distance d , and $d \ll L$. A dielectric slab with dimensions $L \times L \times d$ can slide between the plates. The dielectric slab with dielectric constant ϵ is inserted between the plates a distance x and held there. See side and top view figures. The metal plates are charged to a potential difference V and then disconnected from the voltage source.

Find the electrical force on the slab. Be careful and explicit about its direction.



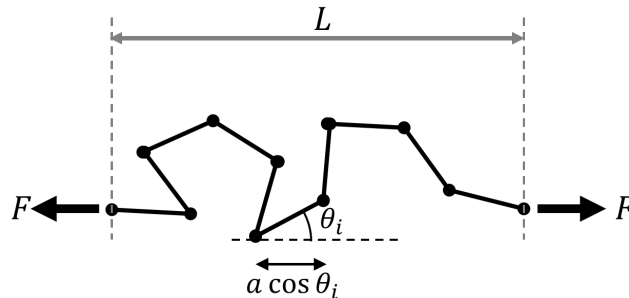
4. EM-2

- (a) (5 points) Twelve identical resistors, 1 Ohm each, are connected to form a cube. What's the reading of an ohmmeter connected to the two opposite vertices of the cube, A and B (see figure)?
- (b) (5 points) How would the answer change if the Ohmmeter is connected across one of the cube sides, i.e., across points A and C in the figure?



5. SM-1

A very crude model of an elastic band is a linear chain of $N \gg 1$ rigid rods each of length a joined end to end by frictionless hinges which allow adjacent rods to be in any relative orientation in the three dimensional space with no energy cost. A force F is applied in opposite directions to the two ends of the chain, as sketched below.



The i^{th} rod makes an angle θ_i with the applied force F , so that the total length of the chain is

$$L = a \sum_{i=1}^N \cos \theta_i.$$

(a) (3 points) Find the partition function $Z(T, F, N)$, where T is the temperature.

(b) (3 points) Show that the mean length of the chain is

$$\langle L \rangle = Na \left(\coth \frac{Fa}{k_B T} - \frac{k_B T}{Fa} \right).$$

(c) (2 points) Find the mean length $\langle L \rangle$ when (i) $Fa/k_B T \gg 1$ and (ii) $Fa/k_B T \ll 1$.

(d) (2 points) Your expression in part (c) says that $\langle L \rangle$ decreases as T increases. Try to give a physical explanation of this apparent contradiction as most real materials expand when heated.

6. SM-2

The excitations of the vibrational modes of an elastic solid are called phonons, and obey Bose-Einstein statistics. In a particular three-dimensional solid of volume V , the phonon energy ϵ is related to its frequency ω by

$$\epsilon = \hbar\omega.$$

The density of phonon states $g(\omega)$ is given by

$$g(\omega) = \frac{V\omega^2}{2\pi^2c_s^3},$$

where c_s is the speed of sound in the solid. There are three degenerate modes at each ω , two transverse and one longitudinal. The maximum vibration frequency is ω_{max} , and it corresponds to neighboring atoms in the solid oscillating out of phase. c_s , ω_{max} , V , and the total number of atoms in the solid, N , are related by

$$c_s = \omega_{max} \left(\frac{6\pi^2 N}{V} \right)^{1/3}.$$

- (a) (4 points) Write an expression for the total vibrational energy of the solid $\langle E \rangle$ at temperature T . Your expression for $\langle E \rangle$ can be left as an integral over ω .
- (b) (3 points) Find the leading order behavior of the heat capacity C_V in the low temperature limit, i.e. when $k_B T \ll \hbar\omega_{max}$. Write your answer in terms of k_B , T , \hbar , ω_{max} and N . You may find the following integral useful: $\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$.
- (c) (3 points) Find the leading order behavior of the heat capacity C_V in the high temperature limit, i.e. when $k_B T \gg \hbar\omega_{max}$. Write your answer in terms of the same parameters as in part b.

7. QM-1

A two-level atom described by the Hamiltonian \hat{H}_0 has eigenstates $|1\rangle$ and $|2\rangle$ with energy separation $\hbar\omega_{21} \equiv E_2 - E_1$. The atom is initially in its ground state $|1\rangle$ and at time $t \geq 0$ it is illuminated with an electric field $\varepsilon = \varepsilon_0(e^{i\omega t} + e^{-i\omega t})$ in the x direction. The electric field oscillates at frequency ω and has amplitude $|\varepsilon_0|$.

- (a) (2 points) Write down the Hamiltonian for time $t \geq 0$ in terms of \hat{H}_0 and a perturbation potential \hat{V} .
- (b) (2 points) Write a general form of the solution of the Hamiltonian defined in part (a), at time $t \geq 0$.
- (c) (3 points) What is the probability that the atom will be in state $|2\rangle$ at time $t \geq 0$, if $|\varepsilon_0|$ is small and $\omega = \omega_{21}$. Assume that for small $|\varepsilon|$ fast oscillating terms of the form $e^{i\omega t}$ can be neglected. Express your answer in terms of $W_{12} \equiv |\varepsilon_0|\langle 1|\hat{x}|2\rangle = W_{21} = |\varepsilon_0|\langle 2|\hat{x}|1\rangle$.
- (d) (3 points) How is your result in (c) modified if ω is slightly detuned from ω_{21} ? That is, what is the probability that the atom will be in state $|2\rangle$ at time $t \geq 0$ if $\omega_{21} \neq \omega$?

8. QM-2

Consider a system with only two energy levels (like a spin-1/2 particle, e.g, but you don't need to think explicitly about spin). The unperturbed nondegenerate energy levels are E_1^0 and E_2^0 (assume that $E_2^0 > E_1^0$). A time-independent perturbation λV is added to this system. In terms of the eigenstates $|1^0\rangle$ and $|2^0\rangle$ of the unperturbed Hamiltonian, the matrix elements of the perturbation are: $\langle 1^0|\lambda V|2^0\rangle = \langle 2^0|\lambda V|1^0\rangle = \lambda\Delta$; the diagonal elements of the perturbation are zero.

- (a) (4 points) Find the **exact** energy levels of the full Hamiltonian, unperturbed plus λV , by finding the eigenvalues E of $H|\psi\rangle = E|\psi\rangle$.
- (b) (6 points) Using perturbation theory calculate both the first order and second order corrections (in λ) to the unperturbed energy levels E_1^0 and E_2^0 . Check that to this order in λ the perturbation theory calculation agrees with the results of part a.

9. QM-3

“Phase states” for the harmonic oscillator. In this problem we work with the harmonic oscillator Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2}m\omega^2\hat{x}^2.$$

Let M be some fixed positive integer. For each integer k in the range $0, 1, \dots, M-1$, we define an angle $\theta_k = 2\pi\frac{k}{M}$ and a corresponding “phase state” $|\theta_k\rangle$ by the following equation:

$$|\theta_k\rangle = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} e^{-in\theta_k} |n\rangle,$$

where $|n\rangle$ is an energy eigenstate for the harmonic oscillator, i.e. $\hat{H}|n\rangle = (n + \frac{1}{2})\hbar\omega|n\rangle$.

(a) (4 points) Compute the inner product $\langle\theta_k|\theta_m\rangle$ of two phase states when $k \neq m$ and $\langle\theta_k|\theta_k\rangle$. In evaluating the sum, you may use the fact that the sum of a geometric series is

$$\sum_{n=0}^{M-1} a^n = \frac{1 - a^M}{1 - a}.$$

(b) (3 points) If our initial state $|\psi(0)\rangle = |\theta_k\rangle$, what is the new state $|\psi(t)\rangle$ at time t ?

(c) (3 points) Compute the expectation value $\langle x \rangle = \langle\psi(t)|\hat{x}|\psi(t)\rangle$ (as a function of time) in the state $|\psi(t)\rangle$ from part (b).

Useful formula for harmonic oscillator: lowering and raising operators are defined as

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p}_x \right), \quad (1)$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p}_x \right), \quad (2)$$

They act on the energy eigenstates of the harmonic oscillator:

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad (3)$$

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle \quad (4)$$

10. QM-4

The Hamiltonian $H = H_0 + \xi \vec{L} \cdot \vec{\sigma}$ describes in a simplified way an electron moving in the electrostatic field of a proton, described by H_0 , and under the influence of a spin-orbit interaction $\xi \vec{L} \cdot \vec{\sigma}$ with ξ : constant (independent of \vec{r}). Let \vec{u} be a unit vector in the $x-z$ plane making an angle θ with the z -axis. Assume that at $t = 0$ the electron is in an eigenstate of L^2 and L_z with eigenvalues $l = 1$ and $m = 1$ and that its spin is pointing in the direction of \vec{u} .

- (a) (4 points) Find the probability that a measurement at time $t = 0$ will show the spin pointing along the z -axis.
- (b) (3 points) Find the probability that at $t = 0$ the component of the total angular momentum along the z -axis is $\pm \frac{1}{2}$.
- (c) (3 points) Answer question (a) for $t > 0$.