DEPARTMENT OF PHYSICS
BROWN UNIVERSITY
Written Qualifying Examination for the Ph.D. Degree
September 6, 2016

READ THESE INSTRUCTIONS CAREFULLY

1. The time allowed to complete the exam is 10:00 AM - 3:00 PM.

2. All work is to be done without the use of books or papers and without help from anyone. The use of calculators or other electronic devices is not permitted.

3. Use a separate answer book for each question, or two books if necessary.

4. DO NOT write your name in your booklets. Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.

5. Write the problem number in the center of the outside front cover. Write nothing else on the inside or outside of the front and back covers. Note that there are separate graders for each question.

6. Answer one (and only one) problem from each of the five pairs of questions. The pairs are labeled as follows:

    Classical Mechanics CM1, CM2

    Electricity and Magnetism EM1, EM2

    Statistical Mechanics SM1, SM2

    Quantum Mechanics QM1, QM2

    Quantum Mechanics QM3, QM4

    Note that there are two pairs of Quantum Mechanics problems. You have to do one problem from each pair.

7. All problems have equal weight.

1 CM1

A disk in a horizontal plane spins about its axis at a constant angular velocity ω. A frictionless ball is shot out from the center of the disk with speed v at time $t = 0$, initially along the $\hat{x}$-axis in the rotating frame of the disk. Its distance from the center of the disk is therefore $r = vt$ and the angle it makes with the disk's $\hat{x}$-axis is $\theta = -\omega t$ because the disk is rotating under the ball. Thus the ball has Cartesian coordinates in the rotating frame of the disk given by:

$$
\begin{aligned}
x(t) &= +vt \; \cos(\omega t) \\
y(t) &= -vt \; \sin(\omega t)
\end{aligned}
$$

(a) (5 points) Working in the (non-inertial) frame of the rotating disk, write down the equations of motion for the ball in Cartesian coordinates. Make no approximations. The following formula may be useful:

$$
\frac{d\vec{r}}{dt}\bigg|_{inertial} = \frac{d\vec{r}}{dt}\bigg|_{rotating} + \vec{\omega} \times \vec{r}
$$

The analogous equation applies to $\vec{v}$.

Hint: You must consider both the Coriolis force, and the centrifugal force.

(b) (5 points) Show that $x(t)$ and $y(t)$ given above solve the equations of motion that you found in part (a), demonstrating that you can work either in the inertial or the rotating disk frames and arrive at the same result.

2 CM2

(10 points) Two particles move in one dimension at the junction of three springs, as shown in the figure. The springs all have unstretched lengths equal to a, and the force constants and masses are as shown. Find the eigenfrequencies and normal modes of the system.
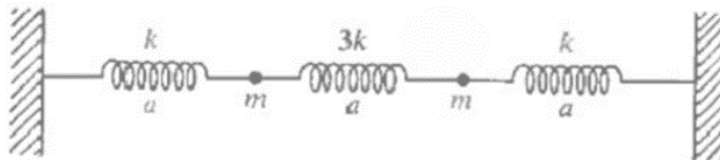
3 EM1

A small disc of radius R carrying a surface charge of density σ lies in the x-y plane with its center positioned at $(x_0, 0, 0)$ in Cartesian coordinates.

(a) (4 points) When the disc rotates with a constant angular speed of ω with respect to its axis (pointing in the negative z direction), calculate the magnetic dipole moment of the disc.

(b) (6 points) An infinitely long line of current I is placed along the y axis. Calculate the force and torque exerted on the spinning charged disc, assuming that $x_0 >> R$.
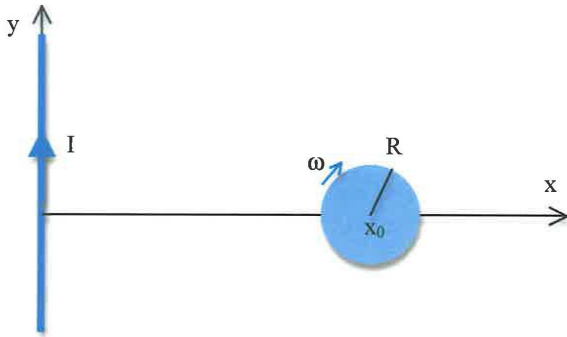
4 EM2

A plane wave of frequency $\nu = \frac{\omega}{2\pi}$ is incident from $x = +\infty$ and traveling in the negative x-direction. The wave is plane polarized along the y-direction.

(a) (2 points) Write down the electric and magnetic fields.

(b) (2 points) If a perfectly conducting plane mirror is introduced at $x = 0$, find the reflected wave.

(c) (3 points) Find the surface currents on the conducting mirror.

(d) (3 points) Calculate the radiation pressure on the conducting mirror.

5   SM1

Consider a system of N non-interacting particles, each fixed in position and carrying a magnetic moment μ, which is immersed in a magnetic field H. Each particle may then exist in one of the two energy states $E = 0$ or $E = 2\mu H$. Treat the particles as distinguishable.

(a) (2 points) The entropy, S, of the system can be written in the form $S = k \ln \Omega(E)$, where k is the Boltzmann constant and E is the total system energy. Explain the meaning of $\Omega(E)$.

(b) (2 points) Write a formula for $S(n)$, where n is the number of particles in the upper state. Crudely sketch $S(n)$.

(c) (2 points) Rewrite the result of (b) using Stirling's approximation for large n: $\ln n! = n \ln n - n$. Find the value of n for which $S(n)$ is maximum.

(d) (2 points) Treating E as continuous, show that this system can have negative absolute temperature.

(e) (2 points) Why is negative temperature possible here but not for a gas in a box?

6   SM2

High energy gamma rays can produce electron-positron pairs under appropriate conditions. Consider a volume of space in thermal equilibrium at temperature T, and assume that electrons, positrons, and photons are in equilibrium with respect to the reaction

$$e^+ + e^- \rightleftharpoons \gamma.$$
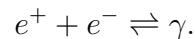
There is an equal number of electrons and positrons, and you may assume the region is enclosed in a chamber of volume V, and that the photons are in equilibrium with the walls.

(a) (3 points) Find the chemical potentials for electrons and for positrons.

(b) (7 points) Find the ratio of the density (number per unit volume) of electrons to the density of photons in the limit $m_e c^2 \gg k_B T$, where m_e is the mass of the electron, c is the speed of light, k_B is Boltzmann's constant, and T is temperature.

Useful:
$$\int_0^\infty \frac{x^2 \mathrm{d}x}{\mathrm{e}^x - 1} \approx 2.4$$

7 QM1

Consider particles of mass m moving in one spatial dimension subject to the potential energy $U(x) = \frac{1}{2}m\omega^2 x^2$ and hence described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \ .$$

The ground state and first excited state are described by the wavefunctions

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right), \qquad \psi_1(x) = \sqrt{\frac{2m\omega}{\hbar}}x\psi_0(x).$$

You may find the following integrals useful:

$$\int_{-\infty}^{\infty} dz\ e^{-az^2} = \sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} dz\ z^2 e^{-az^2} = \frac{1}{2}\sqrt{\frac{\pi}{a^3}},$$

$$\int_{-\infty}^{\infty} dz\ z^4 e^{-az^2} = \frac{3}{4}\sqrt{\frac{\pi}{a^5}}.$$

We now place a pair of identical, noninteracting, spin-1/2 fermions in this potential, in an energy eigenstate with total energy $E = 2\hbar\omega$. Write the properly normalized (spatial part of the) wavefunction and compute the root mean square expectation value $\sqrt{\langle(x_1 - x_2)^2\rangle}$ of the distance between the two particles, in the case when

(a) (4 points) the system has total spin $S = 0$,

(b) (4 points) the system has total spin $S = 1$.

(c) (2 points) For comparison, write the normalized wavefunction and compute $\sqrt{\langle(x_1 - x_2)^2\rangle}$ for a pair of distinguishable bosons living in this potential.

8 QM2

(10 points) A particle of total energy $E = \frac{\hbar^2\alpha^2}{2m}$ moves in a series of N contiguous one-dimensional regions. The potential in the nth region is $V_n = -(n^2-1)E$, where $n = 1, 2, ..., N$. All regions are of equal width $\frac{\pi}{\alpha}$ except for the first and the last, which are of effectively infinite extent. Calculate two transmission coefficients: one for a particle incident from the side of the first ($n = 1$) region and the other for a particle incident from the side of the last ($n = N$) region.

9 QM3

A particle of charge e is scattered from two charged centers of charge e_1 and e_2 on the x-axis at a distance a and $-a$ from the origin. The incident particle is moving along the x-direction, with initial momentum $\vec{k} = k\hat{x}$. The scattered particle has momentum $\vec{k}'$ and is detected on a screen parallel to the y-z plane and reasonably far from the origin.

(a) (6 points) Use the <u>Born approximation</u> to calculate the scattering amplitude $f(\vec{k}, \vec{k}')$.

(b) (4 points) Evaluate the scattering angles at which the maxima and minima of the cross section are reached.

10 QM4

In this problem you will use first-order perturbation theory to find the effect of the spin-orbit interaction on the 3d energy levels of the hydrogen atom. The spin-orbit interaction $\hat{H}_{so}$ of an electron in a hydrogen atom, in CGS units, is:

$$\hat{H}_{so} = \frac{e^2}{2m_e^2 c^2 r^3} \vec{L} \cdot \vec{S}.$$

Here $\vec{L}$ is the orbital angular momentum operator, and $\vec{S}$ is the electron spin operator.

(a) (3 points) Calculate the expectation value $\langle r^{-3} \rangle$ for the unnormalized 3d wavefunction:

$$\psi(r, \theta, \phi) = r^2 e^{-r/3a_0} \left(3\cos^2 \theta - 1 \right).$$

Be sure to normalize. Hint:

$$\int_0^\infty r^n e^{-r/a} dr = a^{n+1} n!.$$

(b) (3 points) What are the possible eigenvalues of the operator $\vec{L} \cdot \vec{S}$ for an electron in a d-orbital?

(c) (2 points) Show that the spin-orbit splitting ΔE of the 3d levels is of order $\alpha^4 m_e c^2$, where α is the fine-structure constant. Pay attention to dimensional analysis.

(d) (2 points) Numerically evaluate the spin-orbit splitting ΔE of the 3d levels in electron-volts (eV), for hydrogen. Give a real number here, not an abstract algebraic expression. Show, by explicit cancellation, that the physical units work out properly.

Possibly useful information:
$$m_e = 9.11 \times 10^{-28} \text{g} \qquad e = 1.602 \times 10^{-19} \text{C} = 4.803 \times 10^{10} \text{esu}$$
$$c = 2.998 \times 10^8 \text{ m/s} \qquad 1\text{eV} = 1.602 \times 10^{-19} \text{J} = 1.602 \times 10^{12} \text{erg}$$
$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137} \qquad a_0 = \frac{\hbar^2}{me^2} = 0.529 \text{\o A}$$