DEPARTMENT OF PHYSICS
BROWN UNIVERSITY
Written Qualifying Examination for the Ph.D. Degree
September 4, 2015

READ THESE INSTRUCTIONS CAREFULLY

1. The time allowed to complete the exam is 12:00–5:00 PM.
2. All work is to be done without the use of books or papers and without help from anyone. The use of calculators or other electronic devices is also not permitted.
3. Use a separate answer book for each question, or two books if necessary.
4. **DO NOT write your name in your booklets.** Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.
5. Write the problem number in the center of the outside front cover. **Write nothing else** on the inside or outside of the front and back covers. Note that there are separate graders for each question.
6. Answer one (and **only** one) problem from each of the five pairs of questions. The pairs are labeled as follows:

Classical Mechanics	CM1, CM2
Electricity and Magnetism	EM1, EM2
Statistical Mechanics	SM1, SM2
Quantum Mechanics	QM1, QM2
Quantum Mechanics	QM3, QM4

   Note that there are two pairs of Quantum Mechanics problems. You have to do **one** problem from **each** pair.

7. All problems have equal weight.

1. CM-1

A particle moves on the surface of a smooth infinite cylinder. The cylinder consists of the points (x, y, z) such that $x^2 + y^2 = R^2$. The only forces are reaction forces. The particle starts at point $(R, 0, 0)$ at time $t = 0$ and reaches the point $(0, R, L)$ at time $t = T$. Find all possible speeds of the particle at $t = 0$.

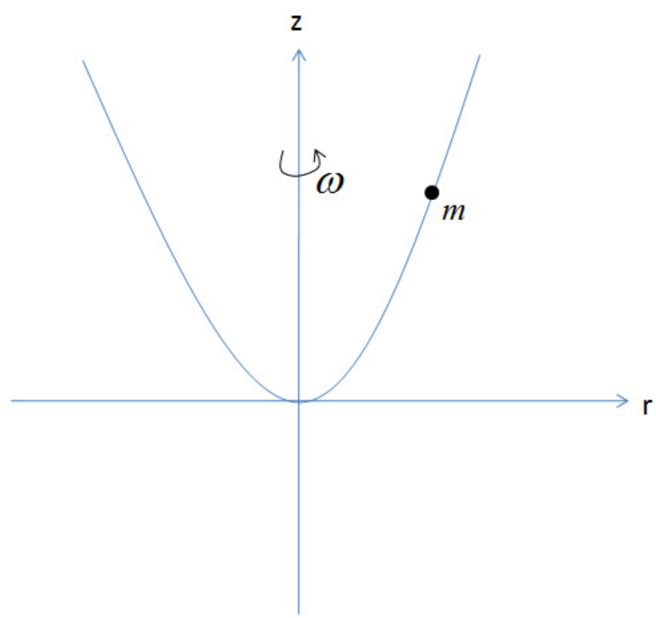
2. CM-2

Consider a bead of mass m sliding without friction on a wire that is bent in the shape of a parabola. The wire parabola is being spun with constant angular velocity ω about its vertical axis (see figure), and the bottom of the parabola is just touching the ground $z = 0$). Let the equation for the parabola be $z = kr^2$.

(a) (3 points) Write down an equation for the Lagrangian for the bead using r as the single generalized coordinate.

(b) (2 points) Write down the equation of motion for the bead.

(c) (5 points) Determine whether there exist any equilibrium positions for the bead (positions where the bead can remain fixed at a given rotational velocity ω). Which of these equilibrium positions are stable?

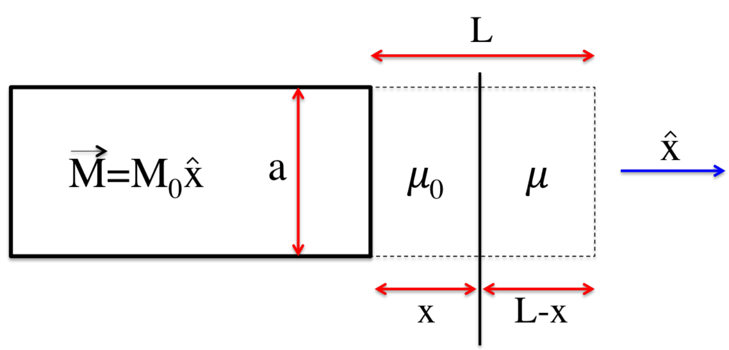


3. EM-1

A very long rectangular bar magnet has a square cross section $a \times a$. It has uniform permanent magnetization $M = M_0\hat{x}$ directed along its long axis, as sketched. The magnet is placed with its square end parallel to and a distance $x \ll a$ from the flat surface of a permeable medium with permeability μ. The permeability of the air in the gap between magnet and medium is μ_0. Consider a small volume $a \times a \times L$ where $x < L \ll a$ as sketched.

(a) Find the magnetic energy $U(x)$ stored in the volume $V = a^2 L$ as sketched.

(b) Find the force F exerted on the magnet by the medium when the distance between the magnet and the permeable medium $x \to 0$ as a function of μ and μ_0. Show that F can be either attractive or repulsive depending on μ/μ_0.

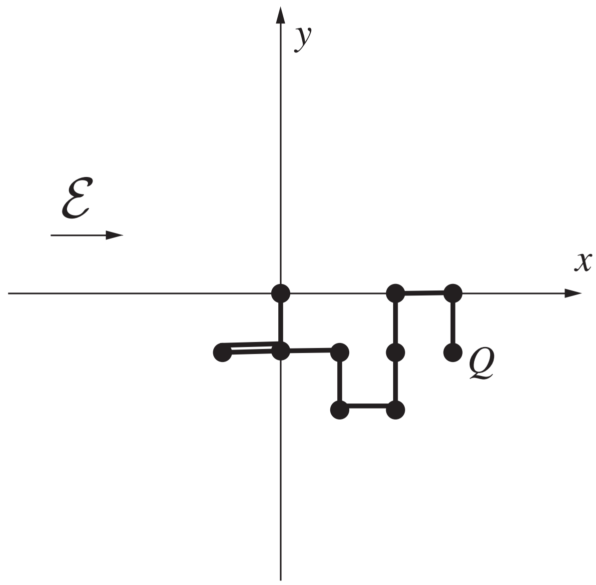


4. EM-2

(a) Determine which two of the three quantities: $\vec{E}^2 + \vec{B}^2, \vec{E} \cdot \vec{B}, \vec{E}^2 - \vec{B}^2$ remain invariant under Lorentz transformations.

(b) Write these two quantities in manifestly covariant form.

5. SM-1



A chain consists of a N links. You may assume $N \gg 1$, but this is not necessary. In any case, N is fixed. Each of the links has length a and may point in the positive or negative x-direction, or in the positive or negative y-direction. A link has energy ϵ. The first link is attached to the origin $(0, 0)$ and a charge Q is attached to the free end of the Nth link. There is an electric field $\mathcal{E}$ pointing in the positive x-direction, so that the energy of a configuration is the internal energy of the links plus the electrostatic energy $-Q\mathcal{E}X$, where X is the x-coordinate of the charge Q. The system is in contact with a heat reservoir at temperature T.

1. (5 points) Calculate $\langle X \rangle$, the average value of X.

2. (5 points) Calculate $\chi = \partial \langle X \rangle / \partial \mathcal{E}$ at $\mathcal{E} = 0$.

6. SM-2

Blackbody Radiation: Consider a closed cubic box with sides of length of L, and walls that absorb light perfectly. These boundary conditions generate a set of discrete energy levels, ϵ_i, for photons inside the box. Furthermore, the box is held at temperature T, so the occupancy of the photon states is determined by thermal equilibrium.

(a) (3 points) Compute the partition function for photons in the box in terms of the discrete energy levels, ϵ_i and the thermal energy, $\beta^{-1} = k_B T$.

(b) (3 points) Show that the average occupancy $\bar{\eta}_s$, of state s is given by the Planck distribution:

$$\bar{\eta}_s = \frac{1}{e^{\beta \epsilon_s} - 1}$$

(c) (4 points) The box is very large in comparison with even the longest relevant wavelength of light. The photon energy levels, ϵ_i, are consequently very closely spaced, as are the possible values of the photon wave-vector, $\vec{k}$, and the photon frequency, ω. It can be shown that the number of photon states (of a single polarization) with magnitude of the wave-vector between k and $k + dk$ per unit volume is:

$$\frac{d^3k}{(2\pi)^3} = \frac{4\pi k^2 dk}{(2\pi)^3}$$

Calculate the average electromagnetic energy density in the box. This corresponds to the mean energy of the photons per unit volume of the box.

Recall that the energy, frequency, and wave-vector of a photon are related by $\epsilon_s = \hbar\omega = \hbar c k$, and that each photon state has two possible polarizations.

You may also find the following useful:

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$$
$$\int_0^\infty \frac{x^2 dx}{e^x - 1} = 2\zeta(3)$$
$$\int_0^\infty \frac{x dx}{e^x - 1} = \frac{\pi^2}{6}$$

7. QM-1

A particle of mass m is initially in the ground state of an infinite one-dimensional square well potential of width L

$$
\begin{aligned}
V &= 0,\ x = 0 \cdots L \\
&= \infty,\ \text{otherwise}
\end{aligned}
$$

From $t = 0$ to $t = T$, the well is perturbed to

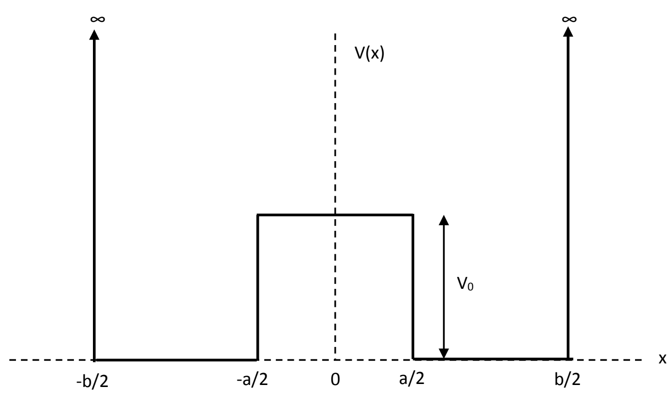
$$
\begin{aligned}
V &= V_0,\ x \in (0, L/2) \\
&= 0,\ x \in (L/2, L) \\
&= \infty,\ \text{otherwise}
\end{aligned}
$$

where $V_0 \ll \hbar^2/mL^2$. Use perturbation theory to find the probability that the particle is in the first excited state after time T.

8. QM-2

(a) (7 points) For the one-dimensional potential shown, assume that $2mV_o a^2 \gg \hbar^2$. Determine the equations governing the energy levels for $E \ll V_o$ up to terms of order $\hbar^2(2mV_o a^2)^{-1}$.

(b) (3 points) Show that the energy levels are almost two-fold degenerate with a splitting controlled by the probability of penetration into the region $-a/2 \le x \le a/2$. Explain this near degeneracy using physical arguments.



9. QM-3

Consider a particle of charge e and mass m in constant $\vec{E}$ and $\vec{B}$ fields.

$$\vec{E} = (0, 0, E)$$
$$\vec{B} = (0, B, 0)$$

(a) (2 points) Write the Schrodinger equation.

(b) (3 points) Separate variables and reduce it to a one-dimensional problem.

(c) (5 points) Calculate the expectation value of the velocity in x-direction in any energy eigenstate.

10. QM-4

The 2005 Nobel Prize in Physics was awarded, in part, for the development of the concept of coherent states. Use the relation $[\hat{a}, \hat{a}^\dagger] = 1$ between lowering and raising operators to answer the following questions about coherent states.

(a) (4 points) Prove that the coherent state vector $|z\rangle \equiv \exp[z\hat{a}^\dagger]|0\rangle$, where z is an arbitrary *complex* number, is an **eigenstate** of the lowering operator $\hat{a}$. What is the corresponding **eigenvalue**? Here, as usual, the state $|0\rangle$ is the ground state which is annihilated by a single application of the lowering operator: $\hat{a}|0\rangle = 0$.

(b) (2 points) Does the raising operator $\hat{a}^\dagger$ also have an eigenstate? If so, write it down. If not, then explain why not.

(c) (2 points) Evaluate $\langle z_1|z_2\rangle$. Use this result to normalize the state $|z\rangle$. Now evaluate the expectation value of the number operator $\hat{N} \equiv \hat{a}^\dagger\hat{a}$:

$$\langle\hat{N}\rangle = \frac{\langle z|\hat{N}|z\rangle}{\langle z|z\rangle}$$

(d) (2 points) The SHO Hamiltonian can be written as: $\hat{H} = \hbar w\hat{a}^\dagger\hat{a}$ if, for simplicity, we drop the additive constant which represents the contribution of zero-point energy. Show that the time-evolution of the state $|z\rangle$ is given by:

$$\hat{U}(t)|z\rangle = |ze^{-iwt}\rangle$$