DEPARTMENT OF PHYSICS
BROWN UNIVERSITY
Written Qualifying Examination for the Ph.D. Degree
August 30, 2013

READ THESE INSTRUCTIONS CAREFULLY

1. The time allowed to complete the exam is 12:00–5:00 PM.
2. All work is to be done without the use of books or papers and without help from anyone. The use of calculators or other electronic devices is also not permitted.
3. Use a separate answer book for each question, or two books if necessary.
4. **DO NOT write your name in your booklets.** Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.
5. Write the problem number in the center of the outside front cover. **Write nothing else** on the inside or outside of the front and back covers. Note that there are separate graders for each question.
6. Answer one (and **only** one) problem from each of the five pairs of questions. The pairs are labeled as follows:

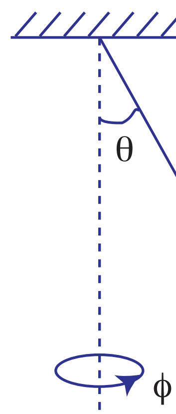
Classical Mechanics	CM1, CM2
Electricity and Magnetism	EM1, EM2
Statistical Mechanics	SM1, SM2
Quantum Mechanics	QM1, QM2
Quantum Mechanics	QM3, QM4

   Note that there are two pairs of Quantum Mechanics problems. You have to do **one** problem from **each** pair.
7. All problems have equal weight.

1. CM-1

Consider a spherical pendulum, i.e., a pendulum whose motion is not restricted to a plane. Using spherical polar coordinates,

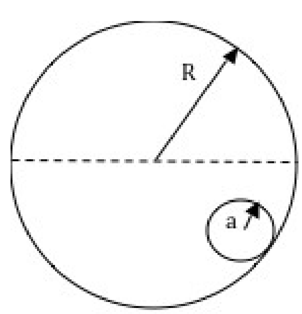
1. Write the Lagrangian and Lagrange's equation of motion.
2. Write the Hamiltonian and Hamilton's equation of motion.
3. Show that the two sets of equations are equivalent and discuss the angular momentum.
4. Discuss the solution for small amplitude (small θ).



2. CM-2

A sphere of radius a and mass m is constrained to roll without slipping on the lower half of the inside surface of a cylinder of radius R, ($R > a$), as shown in the figure.

- Using a Lagrangian find the equation of motion for the center of mass of the sphere (you can use a Lagrange multiplier but it is not essential). The moment of inertia of a sphere through its center is $I = \frac{2}{5}ma^2$.
- Find the frequency of small oscillations of the sphere.

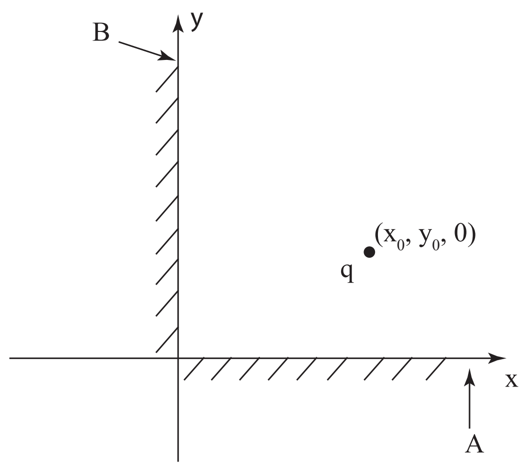


3. EM-1

Two semi-infinite grounded conducting plates intersect on the z-axis. (Out of the paper.) One plate, A, lies on the $x - z$ plane, i.e., points on this plane can be labeled by $(x, 0, z)$, with $-\infty < z < \infty$ and $0 \leq x < \infty$.) The second plate, B, makes a 90^o angle relative to the first plane, (i.e., points on this plate are in the $y - z$ plane, labeled by $(0, y, z)$, $-\infty < z < \infty$ and $0 \leq y < \infty$.

(a) A point charge q is placed at $(x_0, y_0, 0)$, with $x_0 = \sqrt{3}d$, and $y_0 = d$. Find the potential $\phi(x, y, z)$, where $0 < x < \infty$ and $0 < y < \infty$ and z arbitrary. (To simplify all expressions, set $d = 1$.)

(b) Next find the surface charge density on plate A.



4. EM-2

(i) A very long uniform conducting cylinder of radius R carries a current I uniformly distributed over the cross section of the cylinder. Find the magnetic field **H** inside the cylinder at radius $r < R$ from the long axis of the cylinder.

(ii) A cylindrical hole of radius b is bored parallel to the long axis of the cylinder of part (i) with its center a distance a from the axis with $a + b < R$. The current I is uniformly distributed across the conducting part of the cylinder. Find the magnetic field **H** in this hole induced by the current flowing through the conducting part of the long cylinder with the hole. Draw a sketch of the field.

5. SM-1

A quantum particle with spin S is placed in a magnetic field $\mathbf{B}$. The interaction Hamiltonian is $-\mathbf{M} \cdot \mathbf{B}$, where $\mathbf{M} = \mu \mathbf{S}$ is the magnetic moment. Find the magnetic susceptibility at zero field and temperature T.

6. SM-2

A particle of mass m moves in one dimension in the potential

$$\begin{aligned} V(q) &= aq \qquad & q > 0 \\ &= \infty \qquad & 1 < 0 \end{aligned}$$

where a is a positive constant. In this problem assume that classical statistical mechanics can be used.

Suppose that the energy of the particle is known to be in the range between E and $E + dE$, with $dE \ll E$. Use the microcanonical ensemble to calculate:

(a) The probability of finding the particle with position coordinate between q and $q + dq$;

(b) The probability of finding the particle with momentum between p and $p + dp$.

7. QM-1

An uniform electric field along the z direction is applied to a Hydrogen atom.

(a) Treating a uniform, external electric field as a perturbation, evaluate the change in energy of the ground state up to second order in the electric field. Give a physical interpretation of your final result in terms of dipole moments (permanent or induced) and atomic polarizability. (You do not have to evaluate any integrals, just simplify the final expression as much possible.)

(b) Standard perturbation theory does not work for the first excited states with $n = 2$. Explain the reason for this and indicate where the perturbation theory breaks down. Describe how you can avoid these difficulties by applying degenerate perturbation theory to evaluate the first order change in the energy of the $n = 2$ states due to the external electric field. Use symmetry arguments to simplify your results as much as possible. Again, interpret your results in terms of the electric dipoles (induced or permanent).

You may find the following useful:

Some spherical harmonics: $Y_0^0 = (4\pi)^{-1/2}$, $Y_1^{\pm 1} = \mp(3/8\pi)^{1/2} \sin\theta e^{\pm i\phi}$, $Y_1^0 = (3/4\pi)^{1/2} \cos\theta$

Hydrogen radial wavefunctions:

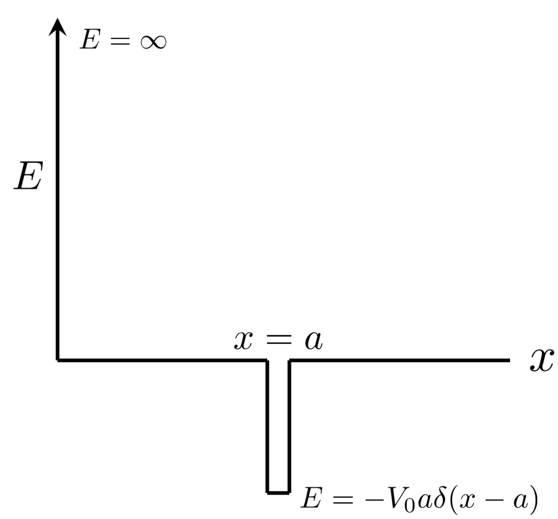
$$R_{10} = \left(\frac{4}{a_0^3}\right)^{1/2} e^{-r/a_0}, \; R_{20} = \frac{1}{\sqrt{2}} a_0^{-3/2} \left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0}, \; R_{21} = \frac{1}{\sqrt{24}} a_0^{-3/2} \frac{r}{a_0} e^{-r/2a_0}.$$

8. QM-2

As shown in the figure, a particle of mass m is confined to the right half-space, in one dimension, by an infinite potential at the origin. There is also an attractive delta function potential $V(x) = -V_0 a\delta(x - a)$, where $a > 0$.

(a) Find an expression for the energy of the bound state.

(b) What is the minimum value of V_0 required for a bound state?



9. QM-3

In this problem you will use first-order perturbation theory to find the effect of the spin-orbit interaction on the $3d$ energy levels of the hydrogen atom. The spin-orbit interaction $\hat{H}_{so}$ of an electron in a hydrogen atom, in CGS units, is:

$$\hat{H}_{so} = \frac{e^2}{2m_e^2c^2}\,\frac{1}{r^3}\,\vec{L}\cdot\vec{S}\ .$$

Here $\vec{L}$ is the orbital angular momentum operator, and $\vec{S}$ is the electron spin operator.

(a) (3 points) To start, calculate the expectation value $\langle r^{-3}\rangle$ for the unnormalized $3d$ wavefunction:

$$\psi(r,\theta,\phi) = r^2 e^{-r/3a_0}(3\cos^2\theta - 1)\ .$$

Be sure to normalize. Hint:

$$\int_0^\infty r^n e^{-r/a} dr = a^{n+1} n!\ .$$

(b) (3 points) What are the possible eigenvalues of the operator $\vec{L}\cdot\vec{S}$ for an electron in a d-orbital?

(c) (2 points) Show that the spin-orbit splitting ΔE of the $3d$ levels is of order $\alpha^4 m_e c^2$ where α is the fine-structure constant. Pay attention to dimensional analysis.

(d) (2 points) Finally, numerically evaluate the spin-orbit splitting ΔE of the $3d$ levels in electron-volts (eV), for hydrogen. I'm looking for a real number here, not an abstract algebraic expression. Show, by explicit cancellation, that the physical units work out properly.

Possibly Useful Information

$$\begin{aligned}
m_e &= 9.11\times10^{-28}\text{g} \\
e &= 1.602\times10^{-19}\text{C} = 4.803\times10^{-10}\text{esu} \\
c &= 2.998\times10^{8}\text{m/s} \\
1eV &= 1.602\times10^{-19}\text{J} = 1.602\times10^{-12}\text{erg} \\
\alpha &\equiv \frac{e^2}{\hbar c} \approx 1/137 \\
a_0 &= \frac{\hbar^2}{me^2} = 0.529\text{Å}
\end{aligned}$$

10. QM-4

Consider a one-dimensional simple harmonic oscillator with mass m and frequency ω. The hamiltonian is

$$\hat{H} = \frac{1}{2}(\hat{X}^2 + \hat{P}^2),$$

where $\hat{X}$ and $\hat{P}$ are the position and momentum operators which can be expressed in terms of the annihilation and creation operators as

$$\hat{X} = \frac{1}{\sqrt{2}}(a^\dagger + a), \quad \hat{P} = \frac{i}{\sqrt{2}}(a^\dagger - a)$$

At $t = 0$ the system is prepared in a superposition of the ground and second excited states,

$$|\psi(0)\rangle = \sqrt{\frac{2}{3}}|0\rangle + \frac{i}{\sqrt{3}}|2\rangle,$$

where $|n\rangle$ are the energy eigenstates or "number states" of the harmonic oscillator.

(a) (2 points) If the number operator is $N = a^\dagger a$, where a and $a^\dagger$ are the creation and annihilation operators respectively, what is the average number of excitations in the system $\langle\hat{N}\rangle$, and what is the uncertainty ΔN to this number?

(b) (2 points) How do $\langle\hat{N}\rangle$ and ΔN change with time?

(c) (2 points) What are the average position $\langle\hat{X}\rangle$ and momentum $\langle\hat{P}\rangle$ of the particle as a function of time? Note that $\hat{X} = \sqrt{m\omega/\hbar}X$ and $\hat{P} = P/\sqrt{m\omega\hbar}$.

(d) (3 points) What are the uncertainties in position ΔX and momentum ΔP as a function of time? Is this a minimum uncertainty wave packet?

(e) (1 point) Sketch a cartoon figure of the position probability as a function of time.